Quadrilateral Geometry

MA 341 - Topics in Geometry
Lecture 19
Varignon's Theorem I

The quadrilateral formed by joining the midpoints of consecutive sides of any quadrilateral is a parallelogram.

PQRS is a parallelogram.
Proof

$PQ \parallel BD$

$RS \parallel BD$

$PQ \parallel RS.$
Proof

\[ QR \parallel AC \]
\[ PS \parallel AC \]
\[ QR \parallel PS. \]
Proof

PQRS is a parallelogram.
Starting with any quadrilateral gives us a parallelogram

What type of quadrilateral will give us
- a square?
- a rhombus?
- a rectangle?
Varignon’s Corollary: Rectangle

The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are perpendicular is a rectangle.

PQRS is a parallelogram
Each side is parallel to one of the diagonals
Diagonals perpendicular ⇒ sides of parallelogram are perpendicular
⇒ parallelogram is a rectangle.
Varignon’s Corollary: Rhombus

The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are congruent is a rhombus.

PQRS is a parallelogram
Each side is half of one of the diagonals
Diagonals congruent $\Rightarrow$ sides of parallelogram are congruent
Varignon’s Corollary: Square

The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are congruent and perpendicular is a square.
Quadrilateral Centers

Each quadrilateral gives rise to 4 triangles using the diagonals.
P and Q = centroids of $\triangle ABD$ and $\triangle CDB$
R and S = centroids of $\triangle ABC$ and $\triangle ADC$
The point of intersection of the segments PQ and RS is the centroid of ABCD
Centroid
Quadrilateral Centers

The centerpoint of a quadrilateral is the point of intersection of the two segments joining the midpoints of opposite sides of the quadrilateral. Let us call this point $O$. 
Centerpoint
Centroid & Centerpoint

Diagram of a quadrilateral with points A, B, C, D, and the centroids G, O, O.
Theorem

The segments joining the midpoints of the opposite sides of any quadrilateral bisect each other.

Proof:
Theorem

The segment joining the midpoints of the diagonals of a quadrilateral is bisected by the centerpoint.
Proof

Need to show that PMRN a parallelogram

In $\triangle ADC$, PN a midline and $PN \parallel DC$ and $PN = \frac{1}{2} DC$

In $\triangle BDC$, MR a midline and $MR \parallel DC$ and $MR = \frac{1}{2} DC$

$\Rightarrow MR \parallel PN$ and $MR = PN$

$\Rightarrow$ PMRN a parallelogram

Diagonals bisect one another.

Then MN intersects PR at its midpoint, which we know is $O$. 

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Theorem

Consider a quadrilateral $ABCD$ and let $E, F, G, H$ be the centroids of the triangles $\triangle ABC, \triangle BCD, \triangle ACD$, and $\triangle ABD$.

1. $EF \parallel AD, FG \parallel AB, GH \parallel BC$, and $EH \parallel CD$;
2. $K_{ABCD} = 9 K_{EFGH}$. 

Diagram of a quadrilateral $ABCD$ with centroids $E, F, G, H$ and parallel segments.
Proof

$M_{BC} =$ midpoint of $BC$. Then $AM_{BC} =$ median of $\triangle ABC$ and $E$ lies $2/3$ of the way between $A$ and $M_{BC}$, $EM_{BC} = 1/3 \ AM_{BC}$.

$DM_{BC} =$ median of $\triangle DCB$ and $DF:FM_{BC} = 2:1 \implies$ in $\triangle ADM_{BC}$ we have $EF \parallel AD$ and $EF = 1/3 \ AD$.

Also

$FG \parallel AB$ and $FG = 1/3 \ AB$

$GH \parallel BC$ and $GH = 1/3 \ BC$

$EH \parallel CD$ and $EH = 1/3 \ CD$

Thus, $EFGH \sim ADCB$ with similarity constant $1/3$. Therefore,

$K_{ABCD} = 9K_{EFGH}$. 

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Theorem

The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$$
Proof

By Law of Cosines $\triangle ABE$

$AB^2 = BE^2 + AE^2 - 2AE \cdot BE \cos(BEA)$

Note that $\cos BEA = FE/BE$, so

$AB^2 = BE^2 + AE^2 - 2AE \cdot FE$

Apply Stewart's Theorem to $\triangle EBC$ we have

$BC^2 = BE^2 + EC^2 + 2EC \cdot FE$

$ABCD$ parallelogram $\Rightarrow$ diagonals bisect each other

Thus $AE = EC$. Adding the first two equations we get

$AB^2 + BC^2 = 2BE^2 + 2AE^2$

Apply this same process to $\triangle CAD$ and we have

$CD^2 + AD^2 = 2DE^2 + 2CE^2$. 
Proof

Now, add these equations and recall that $AE=EC$ and $BE=ED$.

\[
AB^2 + BC^2 + CD^2 + AD^2 = BE^2 + 2AE^2 + 2DE^2 + 2CE^2
\]
\[
= 4AE^2 + 4BE^2
\]
\[
= (2AE)^2 + (2BE)^2
\]
\[
= AC^2 + BD^2
\]
Varignon's Theorem II

The area of the Varignon parallelogram is half that of the corresponding quadrilateral, and the perimeter of the parallelogram is equal to the sum of the diagonals of the original quadrilateral.
Proof

Recall $SP = \text{midline of } \triangle ABD$ and $K_{ASP} = \frac{1}{4} K_{ABD}$

$K_{DSR} = \frac{1}{4} K_{DAC}$

$K_{CQR} = \frac{1}{4} K_{CBD}$

$K_{BPQ} = \frac{1}{4} K_{BAC}$

Therefore,

$K_{ASP} + K_{DSR} + K_{CQR} + K_{BPQ} = \frac{1}{4}(K_{ABD} + K_{CBD}) + \frac{1}{4}(K_{DAC} + K_{BAC})$

$= \frac{1}{4} K_{ABCD} + \frac{1}{4} K_{ABCD}$

$= \frac{1}{2} K_{ABCD}$
Proof

Then,
\[ K_{PQRS} = K_{ABCD} - (K_{ASP} + K_{DSR} + K_{CQR} + K_{BPQ}) \]
\[ = K_{ABCD} - \frac{1}{2} K_{ABCD} \]
\[ = \frac{1}{2} K_{ABCD} \]

Also PQ = \( \frac{1}{2} \) AC = SR and SP = \( \frac{1}{2} \) BD = QR

Easy to see that the perimeter of the Varignon parallelogram is the sum of the diagonals.
Wittenbauer’s Theorem

Given a quadrilateral ABCD a parallelogram is formed by dividing the sides of a quadrilateral into three equal parts, and connecting and extending adjacent points on either side of each vertex. Its area is 8/9 of the quadrilateral. The centroid of ABCD is the center of Wittenbauer’s parallelogram (intersection of the diagonals).