Ceva’s Theorem

MA 341 - Topics in Geometry
Lecture 11
Ceva’s Theorem

The three lines containing the vertices A, B, and C of \( \triangle ABC \) and intersecting opposite sides at points L, M, and N, respectively, are concurrent if and only if

\[
\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1
\]
Ceva’s Theorem

\[
\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}
\]

\[
\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}
\]
Ceva’s Theorem

\[
\frac{BL}{LC} = \frac{K(\triangle ABL) - K(\triangle PBL)}{K(\triangle ACL) - K(\triangle PCL)} = \frac{K(\triangle ABP)}{K(\triangle ACP)}
\]
Ceva’s Theorem

\[
\frac{CM}{MA} = \frac{K(\triangle BMC) - K(\triangle PMC)}{K(\triangle BMA) - K(\triangle PMA)} = \frac{K(\triangle BCP)}{K(\triangle BAP)}
\]
Ceva’s Theorem

\[
\frac{AN}{NB} = \frac{K(\triangle ACN) - K(\triangle APN)}{K(\triangle BCN) - K(\triangle BPN)} = \frac{K(\triangle ACP)}{K(\triangle BCP)}
\]
Ceva’s Theorem

\[
\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = \frac{K(\triangle ACP)}{K(\triangle BCP)} \cdot \frac{K(\triangle ABP)}{K(\triangle ACP)} \cdot \frac{K(\triangle BCP)}{K(\triangle ABP)} = 1
\]
Ceva's Theorem

Now assume that

\[
\frac{AN \cdot BL \cdot CM}{NB \cdot LC \cdot MA} = 1
\]

Let BM and AL intersect at P and construct CP intersecting AB at N', N' different from N.
Ceva’s Theorem

Then $AL$, $BM$, and $CN'$ are concurrent and

$$\frac{AN'}{N'B} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$$

From our hypothesis it follows that

$$\frac{AN'}{N'B} = \frac{AN}{NB}$$

So $N$ and $N'$ must coincide.
Medians

In \( \triangle ABC \), let \( M, N, \) and \( P \) be midpoints of \( AB, BC, AC \).

Medians: \( CM, AN, BP \)

**Theorem:** In any triangle the three medians meet in a single point, called the centroid.

\[
\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = 1
\]

By Ceva’s Theorem they are concurrent.
Orthocenter

Let ΔABC be a triangle and let P, Q, and R be the feet of A, B, and C on the opposite sides. AP, BQ, and CR are the altitudes of ΔABC.

Theorem: The altitudes of a triangle ΔABC meet in a single point, called the orthocenter, H.
Orthocenter

By AA

\( \triangle BRC \sim \triangle BPA \) (a right angle and \( \angle B \))

\( \Rightarrow BR/BP=BC/BA \)

\( \triangle AQB \sim \triangle ARC \) (a right angle and \( \angle A \))

\( \Rightarrow AQ/AR=AB/AC \)

\( \triangle CPA \sim \triangle CQB \) (a right angle and \( \angle C \))

\( \Rightarrow CP/CQ=AC/BC \)

\( \frac{BR}{BP} \cdot \frac{AQ}{AR} \cdot \frac{CP}{CQ} = \frac{BC}{AB} \cdot \frac{AB}{AC} \cdot \frac{AC}{BC} = 1 \)
Orthocenter

By Ceva’s Theorem, the altitudes meet at a single point.
Orthocenter

Traditional route:
BQ intersects AP.
Now construct CH and let it intersect AB at R.
Prove $\triangle ARC \sim \triangle AQB$ making $\angle R = 90$. 

![Orthocenter Diagram](image-url)
Incenter

Let $\triangle ABC$ be a triangle and let $AP$, $BQ$, and $CR$ be the angle bisectors of $\angle A$, $\angle B$, and $\angle C$.

Angle Bisector Theorem: If $AD$ is the angle bisector of $\angle A$ with $D$ on $BC$, then

$$\frac{AB}{AC} = \frac{BD}{CD}$$
Incenter

Proof: Want to use similarity. Where is similarity?

Construct line through C parallel to AB
Incenter

Proof: Want to use similarity. Where is similarity?

Construct line through $C$ parallel to $AB$

Extend $AD$ to meet parallel line through $C$ at point $E$. 
Incenter

\[ \angle BAE \cong \angle CEA - \text{Alt Int Angles} \]
\[ \angle BDA \cong \angle CDE - \text{vertical angles} \]
\[ \triangle BAD \sim \triangle CDE - \text{AA} \]

Therefore

\[ \frac{AB}{CE} = \frac{BD}{CD} \]

Note that \[ \angle CEA \cong \angle BAE \cong \angle CAE \]
\[ \Rightarrow \triangle ACE \text{ isosceles} \Rightarrow CE = AC \text{ and} \]

\[ \frac{AB}{AC} = \frac{BD}{CD} \]
Incenter

Let $\triangle ABC$ be a triangle and let $AP$, $BQ$, and $CR$ be the angle bisectors of $\angle A$, $\angle B$, and $\angle C$.

Theorem: The angle bisectors of a triangle $\triangle ABC$ meet in a single point, called the incenter, $I$. 
Proof: Angle bisector means:

\[
\frac{AB}{AC} = \frac{BP}{PC} \quad \frac{BA}{BC} = \frac{AQ}{QC} \quad \frac{CA}{CB} = \frac{AR}{RB}
\]

By Ceva’s Theorem we need to find the product:

\[
\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA}
\]
Thus by Ceva’s Theorem the angle bisectors are concurrent.
Circumcenter & Perp Bisectors

Does Ceva’s Theorem apply to perpendicular bisectors?
Circumcenter & Perp Bisectors

How can we get Ceva’s Theorem to apply to perpendicular bisectors?
Circumcenter & Perp Bisectors

Draw in midsegments

$EF \parallel BC \Rightarrow$ perpendicular bisector of $BC$ is perpendicular to $EF \Rightarrow$ is an altitude of $\triangle DEF$
Circumcenter & Perp Bisectors

Perpendicular bisectors of $AB$, $BC$ and $AC$ are altitudes of $\triangle DEF$.

Altitudes meet in a single point $\implies$ perpendicular bisectors are concurrent.
Circumcircle

Theorem: There is exactly one circle through any three non-collinear points.

The circle = the circumcircle
The center = the circumcenter, $O$.
The radius = the circumradius, $R$.

Theorem: The circumcenter is the point of intersection of the three perpendicular bisectors.
Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?
Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?
At one end is point of intersection of angle bisector with circumcircle
The other end is point of intersection of exterior angle bisector with circumcircle.
Extended Law of Sines

Theorem: Given ΔABC with circumradius R, let a, b, and c denote the lengths of the sides opposite angles ∠A, ∠B, and ∠C, respectively. Then

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R
\]
Proof

Three cases:
Proof

Case I: \( \angle A < 90^\circ \)
- BP = diameter
- \( \Rightarrow \Delta BCP \) right triangle
- BP = 2R
- \( \Rightarrow \sin P = \frac{a}{2R} \)
- \( \angle A = \angle P \)
- \( \Rightarrow 2R = \frac{a}{\sin A} \)
Proof

Case II: $\angle A > 90^\circ$

BP = diameter

$\Rightarrow \triangle BCP$ right triangle

BP = 2R

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$
Proof

Case III: \( \angle A = 90^\circ \)
BP = \( a = \text{diameter} \)
BP = 2R
2R = \( a = a/\sin A \)
Theorem: Let $R$ be the circumradius and $K$ be the area of $\Delta ABC$ and let $a$, $b$, and $c$ denote the lengths of the sides as usual. Then $4KR = abc$

$$K = \frac{abc}{4R}$$
Proof

\[ K = \frac{1}{2} ab \sin C \]
\[ 2K = ab \sin C \]
\[ \frac{c}{\sin C} = 2R \]
\[ \sin C = \frac{c}{2R} \]
\[ 2K = \frac{abc}{2R} \]
\[ 4KR = abc \]