1. Prove that \( n^2 < 2^n \) for all \( n \geq 5 \). [Hint: While proving this you might be forced to prove another inequality about \( 2^n \) by a separate induction.]

2. Consider the sequence

\[
0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, \ldots
\]

where each string of zeros has one more zero than the previous. Does this sequence converge or diverge? If it converges to a limit \( L \), prove that it converges to \( L \). If it diverges, prove that it diverges.

3. Suppose \( f: \mathbb{R} \to \mathbb{R} \) and \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \).

(a) Prove that \( f(0) = 0 \).

(b) Prove by induction that \( f(nx) = nf(x) \) for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

Let \( \alpha = f(1) \).

(c) Prove that \( f(x) = \alpha x \) for all \( x \in \mathbb{N} \).

(d) Prove that \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). Conclude that \( f(x) = \alpha x \) for all \( x \in \mathbb{Z} \).

(e) Prove that \( f(\frac{x}{n}) = \frac{f(x)}{n} \) for all \( x \in \mathbb{R} \). Conclude that \( f(x) = \alpha x \) for all \( x \in \mathbb{Q} \).

(f) Suppose in addition that \( f \) is continuous, i.e. that for all \( a \in \mathbb{R} \), \( \lim_{x \to a} f(x) = f(a) \). Prove that \( f(x) = \alpha x \) for all \( x \in \mathbb{R} \). [Remark: You have proved that the only continuous homomorphisms of the additive group \((\mathbb{R}, +)\) are of the form \( f(x) = \alpha x \).]