ASSIGNMENT 6
SOLUTIONS
30-October-2006

1. Prove that \( n^2 < 2^n \) for all \( n \geq 5 \).

First, let’s check it for \( n = 5 \). \( 5^2 = 25 < 32 = 2^5 \), so it is true for \( n = 5 \).

Let us assume that \( n^2 < 2^n \) and we need to show that this is true for \( n + 1 \).

\[
(n + 1)^2 = n^2 + 2n + 1 < 2^n + 2n + 1.
\]

In order to complete the proof, we need to know that \( 2n + 1 < 2^n \) for \( n > 5 \). This is clearly true for \( n = 5 \), so assuming that it is true for \( n \), let’s check it for \( n + 1 \).

\[
2(n + 1) + 1 = 2n + 1 + 2 < 2^n + 2 < 2^n + 2^n = 2^{n+1},
\]

so this is true for all \( n > 5 \). Thus, in our original proof, we now have:

\[
(n + 1)^2 = n^2 + 2n + 1 < 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1};
\]

and we are done.

2. Consider the sequence
\[0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, \ldots\]
where each string of zeros has one more zero than the previous. Does this sequence converge or diverge?

If it converges to a limit \( L \), prove that it converges to \( L \). If it diverges, prove that it diverges.

The sequence diverges. One way of showing this is that there is a subsequence which does not converge to 0. Another way of showing this is that regardless of what \( \epsilon \) is given, for any \( N > 0 \) there is another \( a_n = 1 \) for \( n > N \). This will keep the sequence from converging.

3. Suppose \( f : \mathbb{R} \to \mathbb{R} \) and \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \).

(a) Prove that \( f(0) = 0 \).

Since \( f(0) = f(0 + 0) = f(0) + f(0) \) we get \( f(0) = 0 \).

(b) Prove by induction that \( f(nx) = nf(x) \) for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

First, we know that it is true for \( n = 1 \) because \( f(1 \cdot x) = f(x) = 1 \cdot f(x) \). Assume this to be true for \( n \). We need to show that \( f((n + 1)x) = (n + 1)f(x) \).

\[
f((n + 1)x) = f(nx + x)
= f(nx) + f(x)
= nf(x) + f(x)
= (n + 1)f(x)
\]

as we wanted.

Let \( \alpha = f(1) \).

(c) Prove that \( f(x) = \alpha x \) for all \( x \in \mathbb{N} \).

Let \( x \in \mathbb{N} \), then \( f(x) = f(x \cdot 1) = x \cdot f(1) \) by our previous part. Thus, \( f(x) = \alpha x \) if \( x \in \mathbb{N} \).

(d) Prove that \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). Conclude that \( f(x) = \alpha x \) for all \( x \in \mathbb{Z} \).

To see that \( f(-x) = -f(x) \), recall that \( 0 = f(0) = f(x - x) = f(x) + f(-x) \) which leads us to \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). Since \( f(x) = \alpha x \) if \( x \in \mathbb{N} \), then if \( x \in \mathbb{Z} \) and \( x < 0 \), then \( x = -k \), where \( k \in \mathbb{N} \). Thus, \( f(x) = f(-k) = -f(k) = -k\alpha = x\alpha = \alpha x \) if \( x \in \mathbb{Z} \).
(e) Prove that \( f\left(\frac{x}{n}\right) = \frac{f(x)}{n} \) for all \( x \in \mathbb{R} \). Conclude that \( f(x) = \alpha x \) for all \( x \in \mathbb{Q} \).

If \( x \in \mathbb{R} \), then
\[
f(x) = f\left(n \cdot \frac{x}{n}\right) = nf\left(\frac{x}{n}\right).
\]
so \( f\left(\frac{x}{n}\right) = \frac{f(x)}{n} \) for all \( x \in \mathbb{R} \). If \( x \in \mathbb{Q} \), then \( x = \frac{p}{q} \) where \( p, q \in \mathbb{Z} \). We know that
\[
f\left(\frac{p}{q}\right) = \frac{f(p)}{q} = \frac{\alpha p}{q} = \alpha \frac{p}{q} = \alpha x.
\]

(f) Suppose in addition that \( f \) is continuous, i.e. that for all \( a \in \mathbb{R} \), \( \lim_{x \to a} f(x) = f(a) \). Prove that \( f(x) = \alpha x \) for all \( x \in \mathbb{R} \).

Let \( \{a_n\} \) be a sequence of rational numbers whose limit is \( x \). Since the rationals are dense in \( \mathbb{R} \) we can choose such a sequence. Then
\[
f(x) = f(\lim a_n) = \lim f(a_n) = \lim \alpha a_n = \alpha \lim a_n = \alpha x.
\]