1. Find the limits of the following sequences

(a) \( a_n = \sqrt{n^2 + 1} - n \).

\[
a_n = \sqrt{n^2 + 1} - n = \frac{1}{\sqrt{n^2 + 1} + n \sqrt{n^2 + 1} - n} = \frac{1}{\sqrt{n^2 + 1} + n}
\]

To find this limit we will multiply both top and bottom by \(1/n\).

\[
\frac{1}{\sqrt{n^2 + 1} + n} = \frac{\frac{1}{n}}{\sqrt{n^2 + 1} + 1} = \frac{\frac{1}{n}}{\sqrt{1 + \frac{1}{n^2}} + 1}
\]

As \( n \to \infty \) the top goes to 0 and the bottom goes to \(1 + 1 = 2\), so the term approaches 0.

\[
\lim_{n \to \infty} a_n = 0.
\]

(b) \( b_n = \sqrt{n^2 + n} - n \).

We will do something similar here. We need to modify the fraction — in a sense we need to “irrationalize” the denominator.

\[
\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n \sqrt{n^2 + n} - n} = \frac{n}{\sqrt{n^2 + n} + n}
\]

So,

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + n} + n} = \lim_{n \to \infty} \frac{\frac{n}{n}}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2}
\]

(c) \( c_n = \sqrt{4n^2 + n} - 2n \). Again, irrationalizing the denominator, we have

\[
\sqrt{4n^2 + n} - 2n = \frac{n \sqrt{4n^2 + n} - 2n}{\sqrt{4n^2 + n} + 2n \sqrt{4n^2 + n} - 2n}
\]

so,

\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{n}{\sqrt{4n^2 + n} + 2n} = \lim_{n \to \infty} \frac{\frac{n}{n}}{\sqrt{4 + \frac{1}{n}} + 2} = \frac{1}{4}
\]
2. Suppose that \( \lim_{n \to \infty} x_n = 3 \), \( \lim_{n \to \infty} y_n = 7 \) and that all \( y_n \) are nonzero. Determine the following limits:

(a) \( \lim_{n \to \infty} (x_n + y_n) \) 
(b) \( \lim_{n \to \infty} \frac{3y_n - x_n}{y_n^2} \)

(a) \( \lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n = 3 + 7 = 10. \)

(b) \[ \lim_{n \to \infty} \frac{3y_n - x_n}{y_n^2} = \frac{3 \lim_{n \to \infty} y_n - \lim_{n \to \infty} x_n}{(\lim_{n \to \infty} y_n)^2} = \frac{3 \cdot 7 - 3}{7^2} = \frac{18}{49}. \]

3. Let \( a_1 = 1 \) and for \( n \geq 1 \) let \( a_{n+1} = \sqrt{a_n + 1} \).

(a) List the first five terms of \( \{a_n\} \).

\[
\begin{align*}
a_1 & = 1 \\
a_2 & = \sqrt{2} \\
a_3 & = \sqrt{1 + \sqrt{2}} \\
a_4 & = \sqrt{1 + \sqrt{1 + \sqrt{2}}} \\
a_5 & = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{2}}}}
\end{align*}
\]

(b) It turns out that \( \{a_n\} \) converges. Assume that this is true and show that the limit is \( \frac{1}{2}(1 + \sqrt{5}) \).

Since \( \{a_n\} \) converges, its limit \( a \) must satisfy the equation \( a = \sqrt{a + 1} \). What this means is that

\[
\begin{align*}
\lim_{n \to \infty} a_{n+1} & = \lim_{n \to \infty} \sqrt{a_n + 1} \\
\lim_{n \to \infty} a_{n+1} & = \sqrt{\lim_{n \to \infty} a_n + 1} \\
a & = \sqrt{a + 1}
\end{align*}
\]

This is only true because the limit exists!. This means

\[
\begin{align*}
a & = \sqrt{a + 1} \\
a^2 & = a + 1 \\
a^2 - a - 1 & = 0
\end{align*}
\]

The roots to this equation are \( r_1 = \frac{1 + \sqrt{5}}{2} \) and \( r_2 = \frac{1 - \sqrt{5}}{2} \). Now, all of the \( a_n \)'s are positive, so the sequence must converge to the positive root, or

\[ \lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}. \]
4. Let \( a_1 = 1 \) and \( a_{n+1} = \frac{1}{3}(a_n + 1) \) for \( n \geq 1 \).

(a) Find \( a_2, a_3, a_4 \) and \( a_5 \).

\[
\begin{align*}
    a_1 &= 1 \\
    a_2 &= \frac{1}{3}(1 + 1) = \frac{2}{3} \\
    a_3 &= \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9} \\
    a_4 &= \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27} \\
    a_5 &= \frac{1}{3}\left(\frac{14}{27} + 1\right) = \frac{41}{81}
\end{align*}
\]

(b) Use induction to show that \( a_n > \frac{1}{2} \) for all \( n \).

**Method I:**

First, \( a_1 = 1 > 1/2 \) so the statement is true for \( n=1 \). Assume that the statement is true for \( n \), i.e. assume that \( a_n > \frac{1}{2} \). We need to show that \( a_{n+1} > \frac{1}{2} \).

\[
\begin{align*}
    a_{n+1} &= \frac{1}{3}(a_n + 1) \\
    a_{n+1} &> \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}
\end{align*}
\]

Thus, we have shown that if it is true for \( n \) it is also true for \( n + 1 \), and the statement is true by induction.

**Method II:**

Checking you have that

\[
\begin{align*}
    a_n &= 2 + 3 + 9 + 27 + \cdots + 3^{n-2} \\
    &= \frac{2 + \sum_{k=1}^{n-2} 3^k}{3^{n-1}} \\
    &= \frac{2 + \frac{3^{n-1} - 1}{3-1}}{3^{n-1}} \\
    &= \frac{1}{2}\left(1 + \frac{1}{3^{n-1}}\right)
\end{align*}
\]

We need to show this is true for all \( n \).

\[
\begin{align*}
    a_{n+1} &= \frac{1}{3}(a_n + 1) = \frac{1}{3}\left(1 + \frac{1}{2}\left(1 + \frac{1}{3^{n-1}}\right)\right) \\
    &= \frac{11}{32}\left(3 + \frac{1}{3^{n-1}}\right) \\
    &= \frac{1}{2}\left(1 + \frac{1}{3^n}\right)
\end{align*}
\]
which is what we needed to show. Now,

\[ a_n = \frac{1}{2} + \frac{1}{2} \frac{1}{3^{n-1}} > \frac{1}{2}. \]

(c) Show that \( \{a_n\} \) is a nonincreasing sequence.

**Method I:**
We need to show that \( a_n > a_{n+1} \) for all \( n \). We will be done if we can show that \( a_n - a_{n+1} > 0 \) for all \( n \), since the first statement clearly follows from the second.

\[
\begin{align*}
  a_n - a_{n+1} &= a_n - \frac{1}{3}(a_n + 1) \\
  a_n - a_{n+1} &= \frac{2}{3}a_n - \frac{1}{3} > 2 \cdot \frac{1}{2} \cdot \frac{1}{3} = 0
\end{align*}
\]

and we have shown what we needed to show.

**Method II:**

\[
\begin{align*}
  a_{n+1} = \frac{1}{2} \left( 1 + \frac{1}{3^n} \right) &< \frac{1}{2} \left( 1 + \frac{1}{3^{n-1}} \right) = a_n,
\end{align*}
\]

so each term is smaller than the previous and the sequence is nonincreasing.

(d) Find \( \lim a_n \).

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2} \left( 1 + \frac{1}{3^{n-1}} \right) = \frac{1}{2}. \]

5. For each of the following sequences find the \( \text{glb}\{a_n\} \), \( \text{lub}\{a_n\} \), \( \lim \sup\{a_n\} \), and \( \lim \inf\{a_n\} \).

(a) \( \{(−1)^n\}_{n=0}^\infty \) Let \( A \) denote the set of values of this sequence. \( A = \{-1, 1\} \). Thus, \( \text{lub} A = 1 \), \( \text{glb} A = -1 \), \( \lim \inf \{a_n\} = -1 \), and \( \lim \sup \{a_n\} = 1 \).

(b) \( \left\{ \frac{1}{n} \right\}_{n=1}^\infty \) Let \( B \) denote the set of values of this sequence.

\[ B = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}. \]

From this we see that \( \text{lub} B = \text{lub}\{a_n\} = 1 \) and \( \text{glb} B = \text{glb}\{a_n\} = 0 \). The “tails” of this sequence are \( A_n = \left\{ \frac{1}{n}, \frac{1}{n+1}, \ldots \right\} \). Then \( u_n = \text{lub} A_n = \frac{1}{n} \) and \( v_n = \text{glb} A_n = 0 \). Then the \( \lim \sup \) and \( \lim \inf \) are

\[
\begin{align*}
  \lim \sup \{a_n\} &= \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0 \\
  \lim \inf \{a_n\} &= \lim_{n \to \infty} v_n = \lim_{n \to \infty} 0 = 0
\end{align*}
\]

(c) \( \{(−1)^n\}_{n=0}^\infty \) Let \( B \) denote the set of values of this sequence.

\[ B = \{0, -1, 2, -3, 4, -5, 6, \ldots \}. \]
From this we see that \( \text{lub} B = \text{lub} \{a_n\} = +\infty \) and \( \text{glb} B = \text{glb} \{a_n\} = -\infty \). The “tails” of this sequence are \( A_n = \{(-1)^n n, (-1)^{n+1} (n+1)\} \). Then \( u_n = \text{lub} A_n = +\infty \) and \( v_n = \text{glb} A_n = -\infty \). Then the \( \text{lim sup} \) and \( \text{lim inf} \) are

\[
\text{lim sup} \{a_n\} = \lim_{n \to \infty} u_n = +\infty \\
\text{lim inf} \{a_n\} = \lim_{n \to \infty} v_n = -\infty
\]

6. Let \( \{a_n\} \) and \( \{b_n\} \) be the following sequences that repeat in cycles of four.

\[
\begin{align*}
\{a_n\} &= \{0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots \} \\
\{b_n\} &= \{2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \ldots \}
\end{align*}
\]

First, let’s find a few items:

\[
\begin{align*}
\{a_n + b_n\} &= \{2, 2, 3, 1, 2, 2, 3, 1, 2, 2, 3, 1, 2, 2, 3, 1, \ldots \} \\
\{a_n b_n\} &= \{0, 1, 2, 0, 0, 1, 2, 0, 0, 1, 2, 0, 0, 1, 2, 0, \ldots \} \\
\text{lim inf } a_n &= 0 \\
\text{lim sup } a_n &= 2 \\
\text{lim inf } b_n &= 0 \\
\text{lim sup } b_n &= 2
\end{align*}
\]

(a) \( \text{lim inf } a_n + \text{lim inf } b_n = 0 + 0 = 0 \)
(b) \( \text{lim inf}(a_n + b_n) = 1 \)
(c) \( \text{lim inf } a_n + \text{lim sup } b_n = 0 + 2 = 2 \)
(d) \( \text{lim sup}(a_n + b_n) = 3 \)
(e) \( \text{lim sup } a_n + \text{lim sup } b_n = 2 + 2 = 4 \)
(f) \( \text{lim inf } a_n b_n = 0 \)
(g) \( \text{lim sup } a_n b_n = 2 \)