1. Using the Trichotomy Law prove that if $a$ and $b$ are real numbers then one and only one of the following is possible: $a < b$, $a = b$, or $a > b$.

Since $a$ and $b$ are real numbers then $a - b$ is a real number. By the Trichotomy Law we know that $a - b < 0$, $a - b = 0$ or $a - b > 0$. These immediately translate into $a < b$, $a = b$ or $a > b$.

2. We define the absolute value of a real number $a$ by

$$|a| = \begin{cases} 
a, & a \geq 0 \\
-a, & a \leq 0
\end{cases}$$

Prove the following:

(a) $|a + b| \leq |a| + |b|$.

We will each of these by cases. The case where either $ab = 0$ is not interesting, so we will leave it. We must have $a < 0$ or $a > 0$ and $b < 0$ or $b > 0$. Thus, we are left with 4 cases to check: (1) $a > 0$ and $b > 0$, (2) $a < 0$ and $b > 0$, (3) $a > 0$ and $b < 0$, and (4) $a < 0$ and $b < 0$.

In case (1) since both $a$ and $b$ are positive, $a + b$ is positive and $|a| = a$, $|b| = b$, and $|a + b| = a + b$. Therefore $|a + b| = a + b = |a| + |b|$ and the statement is true.

In case (2), since $a < 0$, $|a| = -a$. To show that $|a + b| \leq |a| + |b|$ we must show that

$$|a| + |b| - |a + b| \geq 0.$$ 

Either $a + b \leq 0$ or $a + b \geq 0$.

If $a + b \geq 0$

$$|a| + |b| - |a + b| = (-a) + b - (a + b) = -2a > 0 \text{ since } -a > 0$$

If $a + b \leq 0$

$$|a| + |b| - |a + b| = (-a) + b - (-a + b)) = (a) + b + a + b)) = 2b > 0 \text{ since } b > 0$$

Thus $|a + b| \leq |a| + |b|$ in this case.

Case (3) is similar since the roles of $a$ and $b$ are reversed.

Case (4) is similar to Case (1).
(b) \(|xy| = |x| \cdot |y|\).

Here we break the proof up into the same cases: (1) \(x > 0, y > 0\), (2) \(x < 0, y > 0\), (3) \(x > 0, y < 0\), and (4) \(x < 0, y < 0\).

In Case (1) since \(x > 0\) and \(y > 0\), then \(xy > 0\), and it easily follows that \(|xy| = xy = |x| \cdot |y|\).

In Case (2) since \(x < 0\) and \(y > 0\), then \(xy < 0\), and it easily follows that \(|xy| = -(xy) = (-x)y = |x| \cdot |y|\).

In Case (3) since \(x > 0\) and \(y < 0\), then \(xy < 0\), and it easily follows that \(|xy| = -(xy) = x(-y) = |x| \cdot |y|\).

In Case (4) since \(x < 0\) and \(y < 0\), then \(xy > 0\), and it follows that \(|xy| = xy = (-x)(-y) = |x| \cdot |y|\).

(c) \(|\frac{1}{x}| = \frac{1}{|x|}\), if \(x \neq 0\).

Since \(x \neq 0\), we know that \(\frac{1}{x}\) is its multiplicative inverse, so

\[ 1 = |x \cdot \frac{1}{x}| = |x| \cdot \left| \frac{1}{x} \right|. \]

Solving gives us that \(|\frac{1}{x}| = \frac{1}{|x|}\).

(d) \(|\frac{x}{y}| = \frac{|x|}{|y|}\), if \(y \neq 0\).

Use the above again and the fact that \(\frac{x}{y} = x \cdot \frac{1}{y}\):

\[ \left| \frac{x}{y} \right| = \left| x \cdot \frac{1}{y} \right| = |x| \cdot \left| \frac{1}{y} \right| = |x| \cdot \frac{1}{|y|} = \frac{|x|}{|y|}. \]

(e) \(|x - y| \leq |x| + |y|\).

**Solution Method I:** The eloquent solution uses the results of Part 2a to show this:

\[ |x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|, \]

where the inequality comes from 2a.

**Solution Method II:** You can do this one much like the first one. Break it into cases and do them one at a time. Cases: (1) \(x > 0, y > 0\), (2) \(x < 0, y > 0\), (3) \(x > 0, y < 0\), and (4) \(x < 0, y < 0\).

We need to show in each case that \(|x| + |y| - |x - y| \geq 0\).

In Case (1) we have to deal with two cases \(x - y \leq 0\) and \(x - y \geq 0\). If \(x - y \geq 0\), then \(|x - y| = x - y\) and \(|x| + |y| - |x - y| = x + y - (x - y) = 2y > 0\). If \(x - y \leq 0\),
Thus, this is true in Case (1).

Case (2): In this case $|x| = -x$ and $|y| = y$. Again, we have to consider two cases: $x - y \leq 0$ and $x - y \geq 0$. However, note that if $x < 0$ and $y > 0$, it cannot happen that $x - y \geq 0$. So, $x - y \leq 0$, then $|x - y| = -(x - y) = y - x$ and $|x| + |y| - |x - y| = -x + y - (y - x) = 0 \geq 0$. Thus, this is true in Case (2).

Case (3): In this case $|x| = x$ and $|y| = -y$. Again, we have to consider two cases: $x - y \leq 0$ and $x - y \geq 0$. Again, as in Case (2) it is impossible for $x - y \leq 0$. So, $x - y \geq 0$, then $|x - y| = x - y$ and $|x| + |y| - |x - y| = x - y - (x - y) = 0 \geq 0$. Thus, this is true in Case (3).

For Case (4), $|x| = -x$ and $|y| = -y$. If $x - y \geq 0$, then $|x - y| = x - y$ and $|x| + |y| - |x - y| = -x + (-y) - (x - y) = -2x > 0$. If $x - y \leq 0$, then $|x - y| = -(x - y) = y - x$ and $|x| + |y| - |x - y| = -x + (-y) - (y - x) = -2y \geq 0$. Thus, this is true.

(f) $|x| - |y| \leq |x - y|.$

Solution Method I: There is an eloquent solution here as well.

\[
|x| = |x - y + y| \\
\leq |x - y| + |y| \\
|x| - |y| \leq |x - y|
\]

Solution Method II: You can also break it into cases and do them one at a time. Cases: (1) $x > 0$, $y > 0$, (2) $x < 0$, $y > 0$, (3) $x > 0$, $y < 0$, and (4) $x < 0$, $y < 0$.

We need to show in each case that $|x - y| - (|x| - |y|) = |x - y| - |x| + |y| \geq 0$.

In Case (1) we have to deal with two cases $x - y \leq 0$ and $x - y \geq 0$. If $x - y \geq 0$, then $|x - y| = x - y$ and $|x - y| - |x| + |y| = x - y - x + y = 2y > 0$. If $x - y \leq 0$, then $|x - y| = -(x - y) = y - x$ and $|x - y| - |x| + |y| = y - x - x + y = 2(y - x) > 0$. Thus, this is true in Case (1).

Case (2): In this case $|x| = -x$ and $|y| = y$. This time it is possible for $x - y \leq 0$ but impossible for $x - y \geq 0$. If $x - y \leq 0$, then $|x - y| = -(x - y) = y - x$ and $|x - y| - |x| + |y| = y - x + x + y = 2y > 0$. Thus, this is true in Case (2).

Case (3): In this case $|x| = x$ and $|y| = -y$. This time it is possible for $x - y \geq 0$ but impossible for $x - y \leq 0$. If $x - y \geq 0$, then $|x - y| = x - y$ and $|x - y| - |x| + |y| = x - y - x - y = -2y \geq 0$. Thus, this is true in Case (3).

For Case (4), $|x| = -x$ and $|y| = -y$. If $x - y \geq 0$, then $|x - y| = x - y$ and $|x - y| - |x| + |y| = x - y - (-x) + (-y) = 2(x - y) > 0$. If $x - y \leq 0$, then $|x - y| = -(x - y) = y - x$ and $|x - y| - |x| + |y| = y - x - (-x) + (-y) = 0 \geq 0$. Thus, this is true.
3. The fact that \( a^2 \geq 0 \) for all real numbers \( a \) has tremendous implications. The most widely used of all inequalities is the Schwarz inequality:

\[
x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}
\]

Do ONE of the following:

(a) Prove the Schwarz inequality by using \( 2xy \leq x^2 + y^2 \) (how is this derived?) with

\[
x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \quad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}
\]

first for \( i = 1 \) and then for \( i = 2 \).

The first inequality comes from the fact that \( 0 \leq (x - y)^2 = x^2 - 2xy + y^2 \), so \( 2xy \leq x^2 + y^2 \). Thus, doing the algebra

\[
2xy \leq x^2 + y^2 \leq \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right)^2 + \left( \frac{y_1}{\sqrt{y_1^2 + y_2^2}} \right)^2 \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}
\]

and

\[
2xy \leq \left( \frac{y_1}{\sqrt{y_1^2 + y_2^2}} \right)^2 \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}
\]

Adding these

\[
2 \left( \frac{x_1y_1 + x_2y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \right) \leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} = 2
\]

(b) Prove the Schwarz inequality by first proving that

\[
(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.
\]

First,

\[
(x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2.
\]
Now,

\[(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 = x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2 + x_1^2y_2^2 + x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2 + x_2^2y_2^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2)\]

\[(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 \geq (x_1y_1 + x_2y_2)^2\]

Thus,

\[\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \geq x_1y_1 + x_2y_2\]

and we are done.

4. Prove the following formulæ by induction

(a) \[1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\]

First, check it for \(n = 1\) and we have \(1^2 = \frac{1 \times 2 \times 3}{6} = 1\), so it is true for \(n = 1\). Now, assume it is true for \(k\). We must prove that it is true for \(k + 1\).

\[\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2\]

\[= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}\]

which is what we needed, and we are done.

(b) \[1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2\]

First, check it for \(n = 1\) and we have \(1^3 = (1)^2\), so it is true for \(n = 1\). Now, assume it is true for \(k\). We must prove that it is true for \(k + 1\).
\[ 1^3 + 2^3 + \cdots + (k+1)^3 = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + \cdots + k)^2 + (k+1)^3 = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \left( \frac{k+1)(k+2)}{2} \right)^2 = (1 + 2 + \cdots + (k+1))^2 \]

which is what we needed, and we are done.

5. Find a formula for

(a) \[ \sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + 7 + \cdots + (2n - 1) \]

\[ \sum_{k=1}^{2n} k = \sum_{k=1}^{n} (2k - 1) + \sum_{k=1}^{n} 2k = \frac{(2n)(2n+1)}{2} = \sum_{k=1}^{n} (2k - 1) + 2 \sum_{k=1}^{n} k = \sum_{k=1}^{n} (2k - 1) = 2n^2 + n + \frac{n(n+1)}{2} = 2n^2 + n - (n^2 + n) = n^2 \]

(b) \[ \sum_{i=1}^{n} (2i - 1)^2 = 1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n - 1)^2 \]

Solution Method I:

\[ \sum_{k=1}^{2n} k^2 = \sum_{k=1}^{n} (2k - 1)^2 + \sum_{k=1}^{n} (2k)^2 \]
\[ \sum_{k=1}^{n} (2k - 1)^2 = \frac{(2n)(2n+1)(4n+1)}{6} - 4 \sum_{k=1}^{n} k^2 = \frac{(2n)(2n+1)(4n+1)}{6} - 4 \frac{n(n+1)(2n+1)}{6} = \frac{2n(2n+1)(2n-1)}{6} = \frac{4n^3 - n}{3} \]
Solution Method II:

\[ \sum_{k=1}^{n} (2k - 1)^2 = \sum_{k=1}^{n} (4k^2 - 4k + 1) \]
\[ = \sum_{k=1}^{n} 4k^2 - \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1 \]
\[ = 4 \sum_{k=1}^{n} k^2 - 4 \sum_{k=1}^{n} k + n \]
\[ = 4 \left( \frac{n(n+1)(2n+1)}{6} \right) - 4 \left( \frac{n(n+1)}{2} \right) + n \]
\[ = \frac{2n(n+1)(2n+1)}{3} - 2n^2 - n \]
\[ = \frac{n(2n+1)(2n-1)}{3} = \frac{4n^3 - n}{3} \]

6. Use the given method to find:

(a) \( 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 \)

Following the example from the homework sheet we note that \((k + 1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1\). Then proceeding as the example we would have:

\[(n + 1)^4 - 1 = 4 \sum_{k=1}^{n} n^3 + 6 \sum_{k=1}^{n} n^2 + 4 \sum_{k=1}^{n} n + n \]
\[4 \sum_{k=1}^{n} n^3 = (n + 1)^4 - 1 - 6 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} - n \]
\[4 \sum_{k=1}^{n} n^3 = (n + 1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n \]
\[4 \sum_{k=1}^{n} n^3 = n^4 + 2n^3 + n^2 \]
\[\sum_{k=1}^{n} n^3 = \frac{n^2(n+1)^2}{4} \]

(b) \( 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 \)
First, \((k + 1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1\). So,

\[
(n + 1)^5 - 1 = 5 \sum_{k=1}^{n} n^4 + 10 \sum_{k=1}^{n} n^3 + 10 \sum_{k=1}^{n} n^2 + 5 \sum_{k=1}^{n} n + n
\]

\[
5 \sum_{k=1}^{n} n^4 = (n + 1)^5 - 1 - 10 \frac{n^2(n + 1)^2}{4} - 10 \frac{n(n + 1)(2n + 1)}{6} - 5 \frac{n(n + 1)}{2} - n
\]

\[
5 \sum_{k=1}^{n} n^4 = n^5 + 5n^4 + \frac{5n^3}{3} - \frac{n}{6}
\]

\[
5 \sum_{k=1}^{n} n^4 = \frac{n(2n + 1)(n + 1)(3n^2 + 3n - 1)}{6}
\]

\[
\sum_{k=1}^{n} n^4 = \frac{n(2n + 1)(n + 1)(3n^2 + 3n - 1)}{30}
\]

(c) \[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)}
\]

For this one you need to realize that each term is of the form \(\frac{1}{k \cdot (k + 1)}\) and this can be rewritten as \(\frac{1}{k} - \frac{1}{k + 1}\). Thus,

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n + 1}\right)
\]

\[
= 1 - \frac{1}{n + 1} = \frac{n}{n + 1}
\]

This is a classic example of what is known as a **telescoping sum**.