(1) Let $G$ be a finite group and let $P$ be a normal $p$-subgroup of $G$. Show that $P$ is contained in every Sylow $p$-subgroup of $G$.

(2) Determine all groups of order 21 up to isomorphism.

(3) Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be any subgroup of $G$. Prove that $P \cap H$ is the unique Sylow $p$-subgroup of $H$.

(4) Let $G$ be a finite group of composite order $n$ with the property that $G$ has a subgroup of order $k$ for each positive integer $k$ dividing $n$. Prove that $G$ is not simple.

(5) Scavenger Hunt 1: Algebra Prelim June 2004
Let $(G, \cdot)$ be a group with identity element $e$. Suppose that $a \neq e$ is an element of $G$ such that $a^6 = a^{10} = e$. Determine the order of $a$.

(6) Scavenger Hunt 2: J. Fraleigh Section 6 Exercise 44
Let $G$ be a cyclic group with generator $a$, and let $G'$ be a group isomorphic to $G$. If $\phi : G \to G$ is an isomorphism, show that, for every $x \in G$, $\phi(x)$ is completely determined by the value $\phi(a)$. That is, if $\phi : G \to G'$ and $\psi : G \to G'$ are two isomorphisms such that $\phi(a) = \psi(a)$, then $\phi(x) = \psi(x)$ for all $x \in G$.

(7) Scavenger Hunt: D. Dummit Section 1.6 Exercise 22
Let $A$ be an abelian group and fix some $k \in \mathbb{Z}$. Prove that the map $a \mapsto a^k$ is a homomorphism from $A$ to itself. If $k = -1$ prove that this homomorphism is an isomorphism (i.e., is an automorphism of $A$).

(8) Scavenger Hunt: D. Dummit Section 3.2 Exercise 31
Let $N \leq G$ and $N$ is a normal subgroup of $H$, then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of $G$ in which $N$ is normal (i.e., is the join of all subgroups $H$ for which $N \lhd H$).

(9) Let $G$ be a finite group and let $H$ be a normal Sylow $p$-subgroup of $G$. Show that $\alpha(H) = H$ for all automorphisms $\alpha$ of $G$.

(10) Suppose that $G$ is a group of order $p^n$, where $p$ is prime, and $G$ has exactly one subgroup for each divisor of $p^n$. Show that $G$ is cyclic.

(11) Let $H$ be a Sylow $p$-subgroup of $G$. Prove that $H$ is the only Sylow $p$-subgroup of $G$ contained in $N(H)$.
(12) Show that if $G$ is a group of order 168 that has a normal subgroup of order 4, then $G$ has a normal subgroup of order 28.

(13) Prove that there are 45 elements of order 2 in $A_6$.

(14) Let $G$ be an abelian group, $K$ a group and $f: G \to K$ a group homomorphism. Prove that $f(G) \subseteq K$ is an abelian subgroup of $K$.

(15) Prove that $G$ is abelian if and only if the map $f: G \to G$ by $f(g) = g^2$ is a group homomorphism.

(16) Prove that $(\mathbb{Q} \setminus 0, \cdot)$ is not a cyclic group.

(17) Let $K$ be a Sylow $p$-subgroup of $G$ and $N$ a normal subgroup of $G$. Prove that $K \cap N$ is a Sylow $p$-subgroup of $N$.

(18) Prove that there are no simple subgroups of order 30.

(19) Let $K$ be a $p$-Sylow subgroup of $G$ and $N$ a normal subgroup of $G$. If $K$ is a normal subgroup of $N$, prove that $K$ is normal in $G$.

(20) If $K$ is a $p$-Sylow subgroup of $G$ and $H$ is a subgroup that contains $N(K)$, prove that $[G : H] \equiv 1 \mod p$.

(21) How many elements of order 5 does a non-cyclic group of order 55 have?

(22) If $P$ is a Sylow $p$-subgroup of $G$, prove that $P$ is the only Sylow $p$-subgroup of $N(P)$.

(23) Let $G$ be a group of order 105. Show that $G$ has a subgroup of order 35.

(24) If $|G| = pqr$ with $p \leq q \leq r$ primes, prove that $G$ is not simple.

(25) Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2, and in fact there is only one such group.

(26) Prove that there are 28 homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $D_4$.

(27) Prove that for every integer $1 \leq n \leq 59$ there are no non-abelian simple groups of order $n$.

(28) An abelian group has 8192 has elements of the following orders:

<table>
<thead>
<tr>
<th>order</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td># of elements</td>
<td>1</td>
<td>31</td>
<td>224</td>
<td>1792</td>
<td>2048</td>
<td>4096</td>
</tr>
</tbody>
</table>

Determine the isomorphism type of the group.

(29) Show that the multiplicative group $(\mathbb{Z}/2^l\mathbb{Z})^*$ for $l \geq 3$ is a direct product of a cyclic group of order 2 and another cyclic group of order $2^{l-2}$.

To do this, it will help to show that $\{(-1)^a 5^b | a = 0, 1 \text{ and } 0 \leq b < 2^{l-2}\}$ is a reduced residue system mod $2^l$. You may also use the fact that the order of 5 modulo $2^l$ is $2^{l-2}$.

(30) Let $H$ be a proper subgroup of a finite group $G$. Prove the group $G$ is not the union of the conjugate subgroups of $H$.


(31) Prove that any group of order 1365 is not simple.

Source: Jim Brown, homework problem from MA 851 Fall 2010 at Clemson University

(32) Show that there are two isomorphism classes of groups of order 6, the class of the cyclic group with six elements and the class of the symmetric group $S_3$.


(33) Count and give a combinatorial interpretation of the number of abelian groups of order $2^n$ for $n \in \mathbb{N}$. Give a geometric interpretation of the abelian groups of order 8.

(34) Suppose a group $G$ has elements $u$ and $v$ such that $u^m = e, \, uvu^{-1} = v^k$, where $k > 1, \, m > 0$. Prove that $|v|$ is finite.

(35) Let $G$ be a group, and let $f : G \rightarrow G$ be defined by $f(g) = g^2$. Give necessary and sufficient conditions for $f$ to be an automorphism.

(36) Let $G$ be a finite group and let $P$ be a normal $p$-subgroup of $G$. Show that $P$ is contained in every Sylow $p$-subgroup of $G$.

(37) Show that $A_5$ has no subgroup of order 15.

(38) Show that $A_5$ has no subgroup of order 30. (One possible approach to this is showing that every group of order 30 has a subgroup of order 15).

(39) Show that the number of conjugacy classes in $S_n$ is $p(n)$ where $p(n)$ is the number of ways, neglecting the order of the summands, that $n$ can be expressed as a sum of positive integers. The number $p(n)$ is the number of partitions of $n$.

(40) Show that the number of conjugacy classes in $S_n$ is also the number of different abelian groups (up to isomorphism) of order $p^n$, where $p$ is a prime number.

(41) Let $H$ be a normal subgroup of order $p^k$ of a finite group $G$. Show that $H$ is contained in every $p$-Sylow subgroup of $G$.

(42) Let $G$ be a finite group with the property that for each positive integer $n$, the equation $x^n = 1$ has at most $n$ solutions in the group. Prove that $G$ is cyclic.

(43) Show that any finite $p$-group $G$ is isomorphic to a group of upper triangular matrices with ones on the diagonal (unitriangular matrices) over $\mathbb{F}_p$.

A possible approach to this problem follows:
• Take \( n \in \mathbb{N} \) to be given. Use a counting argument to show that the unitriangular group (group of all \( n \times n \) unitriangular matrices) is a \( p \)-Sylow subgroup of the general linear group (group of all invertible \( n \times n \) matrices) over \( \mathbb{F}_p \).
• Note that the symmetric group embeds in the general linear group using permutation matrices.
• Note that \( G \) is isomorphic to a subgroup of a symmetric group.
• Apply the fact that any two \( p \)-Sylow subgroups are conjugate.

(44) Let \( G \) be a group, and let \( \text{Aut}(G) \) be the group of all automorphisms of \( G \) together with the operation of function composition. Suppose that \( G \) is non-Abelian. Show that \( \text{Aut}(G) \) is not cyclic.

(45) Let \( G \) be a group and \( p \) be a prime. Suppose that \( H = \{g^p | g \in G\} \). Show that \( H \) is a normal subgroup of \( G \) and that every nonidentity element of \( G/H \) has order \( p \).

(46) Let \( G \) be an Abelian group. Determine all homomorphisms from \( S_3 \) to \( G \).

(47) Let \( G \) be an Abelian group and let \( n \) be a positive integer. Let \( G_n = \{g \in G | g^n = e\} \) and \( G^n = \{g^n | g \in G\} \). Prove that \( G/G_n \) is isomorphic to \( G^n \).

(48) How many elements of order 5 does a non-cyclic group of order 55 have?

(49) Prove that there are no simple groups of order 120.

(50) Show that every group of order 56 has a proper normal subgroup.

(51) If \( |G| = pqr \), with \( p < q < r \) primes, the \( G \) is not simple.

(52) If \( G/Z(G) \) is cyclic, prove that \( G \) is abelian.

(53) Prove that a non-cyclic group of order 21 must have 14 elements of order 3.

(54) Suppose that the symmetric group \( S_4 \) has a normal subgroup \( H \) of order 4. Show that the quotient group \( S_4/H \) is either cyclic or isomorphic with \( S_3 \). In the later case describe the subgroup \( H \).

(55) Show that there exist non-abelian groups that have the property \((xy)^n = x^ny^n\) for all \( x, y \) in \( G \) and \( n = k, k + 1 \) for some \( k \).

(56) (Algebra Prelim, January 5, 2012). Let \( G \) be a group of order \( 5^2 \cdot 7 \).

(a) Show that \( G \) is abelian.
(b) Show that if $G$ does not contain an element of order 25, then it contains an element of order 35.

(57) (Algebra Prelim, January 7, 2013). Let $G$ be a group of odd order and let $H$ be a normal subgroup of order 5. Show that $H$ is contained in the center of $G$.

[Hint: Consider the centralizer $C_G(H)$ and show that the quotient is isomorphic to a subgroup of $\text{Aut}(H)$.]