Weekly Activities Ma 110

Fall 2008

As of October 27, 2008

We give detailed suggestions of what to learn during each week. This includes a reading assignment as well as a brief description of the main points to understand.

We do not actually list the various formulas to be memorized, since these can be picked up from the text and memorized using the on line collections of formulas.

Week 1

Read: Chapter 1.1, 1.2.

Think about rational and irrational numbers and the idea of the real number line. Appendices 12.1, 12.2, 12.7 are useful.

Appendices 12.3, 12.5 are meant to provide better understanding of power series and helps understand the power series used in the Euler representation of complex numbers.

However, the actual reading and deeper understanding of these concepts from the appendix is a long term project to be carried out over multiple weeks while the course is progressing.

Section 1.1 Learn how to add, multiply and divide complex numbers. Think about how the complex number can be associated with points in a plane and what the Euler representation could mean. These topics will be clarified later.

Section 1.2 Learn the distinction between the terms variable, parameter and indeterminate. Remember that any letter which is not already a fixed number (like π) can be treated as any of these three entities. However, the choice dictates how the calculations are carried out.

Week 2

Read: Chapter 1.3, 1.4, 1.5. The discussion of Section 5 will spill into next week.

Section 1.3 Be sure to understand the concepts of coefficient, degree and exponents of monomials. Carefully study how these depend on the announcement of chosen variables.

Understand that a polynomial (i.e. a sum of monomials) must be simplified and monomials with like exponents collected before applying various definitions or doing further work.

Section 1.4 Learn the definition of the degree of any polynomial and the leading coefficient of a polynomial in one variable. Study what happens to the degree and the leading coefficient when we combine the polynomials by addition or multiplication. This extends to combining several polynomials.

Section 1.5 Several examples of polynomial operations are described here. It is important to learn each in detail.

Ex. 1 Learn the various shortcuts to picking specific terms in a sum or product. These are great time savers and useful to know since sometimes we don't actually know the whole polynomials.

Ex. 2 Further examples of efficient calculations.

Ex. 3 An ordinary integer can be thought of as a polynomial in 10 with coefficients from $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9$. Then the power of polynomial calculations gives interesting results about integer calculations.

Ex. 4 This illustrates the power of deduction.

We experiment with expansion of powers $(x + t)^i$ for i from 1 to 4. With the expansions in hand, we deduce a neat formula for $(x+t)^n$ for any positive integer n. The binomial coefficients ${}_nC_r$ which we develop have uses in many branches of Mathematics and applications.

Ex. 5 We introduce useful new functions called rational functions (ratios of two polynomials) which are similar to rational numbers. It is important to study the additions, multiplications and divisions of rational functions.

We also define the important operation of compositions of polynomials or rational functions which corresponds to substituting $q(x)$ for x in the formula of $p(x)$. This is written as $p(q(x))$. It is important to have a good practice in evaluating this composition.

Ex. 6 Given a quadratic polynomial $q(x) = ax^2 + bx + c$ with $a \neq 0$ we study a simple but useful substituion $q(u + s)$ with the property that the new quadratic polynomial in variable u has no middle term.

It turns out that if we take $s = -\frac{b}{2a}$, then we get

$$q(x) = q(u - \frac{b}{2a}) = au^2 - \frac{b^2 - 4ac}{4a}.$$

This process is called completing the square or killing the linear term.

This simple calculation has many applications to be studied later.

Week 3

Read: Chapter 2.

This chapter gives a complete analysis of solving several linear equations in one or more variables. It represents the start of the subject called Linear Algebra.

Section 2.1 Learn what a linear equation is and how it depends on the choice of variables. Learn the meaning of a solution of a linear equation and how to verify it by substitution. Learn the idea of a system of linear equations and the meaning of their solution. Learn that a system is consistent if it has at least one solution and inconsistent otherwise.

Section 2.2 Note that the solution to $ax = b$ is $x = \frac{b}{a}$, if $a \neq 0$. If $a = 0$ and $b \neq 0$ then there is no solution(or the equation is inconsistent.) If $a = b = 0$ then there are infinitely many solutions.

This illustrates the all important $0, 1, \infty$ principle.

Section 2.3 Several equations in one variable are solved by solving one of them and checking if the solution satisfies the others. If any one of the equations is ionconsistent, then the whole system is inconsistent.

Section 2.4 Now several equations in two variables are solved by solving one of them for some variable and the result substituted in the others. The other equations now yield a system of equations in one variable and we know how to solve these from section 2.3.

Section 2.5 We extend the idea of section 2.4 to a system of several equations in several variables.

Note that the $0, 1, \infty$ principle still holds. However, the appearance of the solution is now different. Finally some of the variables may be declared to be arbitrary (or free) and the others expressed as linear expressions in them. It is possible that there are no free variables and then the solution is unique.

Section 2.6 Having learned a method of solution of a system of equations, we now give several efficient techniques.

- **Elimination.** One way of reducing the number of variables is to add suitable multiples of one equation to others and kill the coefficients of the chosen variables. This avoids the step of explicitly solving one equation for the chosen variable.

- **Cramer's Rule.** Using the notation of determinants, one can just write down explicit formula for solving two equations in two variables. There are simple tests for deducing when there is no solution, single solution or many solutions. The method has the advantage of giving the value of a chosen variable without solving the whole thing and quickly deciding if there is a unique solution, without finding one!

The method generalizes to three equations in three variables and more generally to n equations in n variables. You should consult Appendix 12.6 for more details and pointers to further study.

Week 4

Read: Chapter 3. This chapter contains material which may be very different form anything you might have studied in high school.

If you learn the definitions well and follow instructions, it is actually rather easy and very useful in future Mathematics courses.

Section 3.1 The section generalizes the idea and the use of long division in integers.

Learn the division algorithm and definitions of quotient, remainder, "modulo"and divisibility (with its notation).

Learn the definitions of GCD, LCM. Learn the Euclidean Algorithm for finding the GCD along with the calculator technique to carry out the steps.

Learn and follow the details described in the book.

Note. You might have learned a method for finding GCD which involves factoring the given numbers and then picking up common factors. This method, though correct, is useless for large numbers, since factoring large numbers is a very difficult task.

Section 3.2 We learn a clever extension of the Euclidean Algorithm called the Āryabhaṭa Algorithm. It lets us efficiently write the GCD as a combination of the given integers. It was developed to solve the equation $ax + by = c$ where a, b, c are integers and we want all integer solutions of x, y.

Learn the Āryabhaṭa Algorithm well and its various uses.

Section 3.3-3.5 Here we learn how to carry out similar calculations with polynomials instead of integers. While the process is same, the calculations can get messy since polynomials can have many coefficients.

We will typically work with simple examples or with partially solved examples where the remaining work is not too messy.

It is, however, crucial that you learn all the definitions and the algorithm itself.

Thus, learn the meaning of divisibility, division algorithm, quotient, remainder and long division in polynomials.

Also learn what a monic polynomial is (one with leading coefficient 1) and the definitions of the GCD and LCM where the answer from the algorithm must be made monic by dividing the polynomial by its leading coefficient.

A special skill is to "guess" the right remainder without actually going through the whole division process. This is crucial to learn.

Week 5

This week is mainly spent in reviewing and memorizing important formulas and definitions. Review old homeworks and main calculation techniques.

The first exam takes place on Friday.

Week 6

Read: Chapter 4. In this chapter, we formally learn how coordinate systems are defined and used to study geometric objects: lines, planes, curves etc.

Section 4.1 Learn how points on a line are associated with the real numbers by setting up a suitable origin and a unit point (i.e. a coordinate system).

Learn how to do the same for a plane to associate its points with pairs of real numbers by choosing an origin, an X-axis and a Y-axis (i.e. a coordinate system).

Section 4.2 Learn the distance formula on a line and a plane in terms of the chosen coordinates.

Recall that complex numbers are naturally identified with points in the plane by making their Argand diagrams, so that a point $P(a, b)$ matches with the complex number $a + bi$.

Learn the geometric meaning of complex conjugation (reflection in the X-axis) and absolute value (distance from the origin).

Section 4.3, 4.4 Learn how different coordinate systems on the same line are related. This involves getting new coordinates for each point and the formulas for the new coordinates are linear expressions.

Learn that a similar relation holds for plane coordinates. In the case of a plane, we usually choose rather restricted changes so the geometric figures are not distorted too much (stay similar, except for scale change and flips).

More general coordinate changes in a plane are also discussed but not heavily used.

It is important to view a change of coordinates as a map from line to line or plane to plane. To do this, we imagine that a point with old coordinates is mapped to a point with its new coordinates.

Both these views have uses in applications.

An **isometry** is a map of a line to itself which preserves all distances. Such maps are completely described. Similar calculations are done in a plane.

Study the various examples carefully.

Week 7

Read: Chapter 5. This chapter studies the lines in the plane which is a familiar subject. However, it also introduces an important new idea of describing the same lines in parametric form. This new form has the advantage of making many calculations much simpler. Moreover, it extends well to higher dimensions.

As it turns out, to get the full power of the parametric form, you need to be able to go back and forth between the usual form and the parametric form.

The best way to contrast the two is to observe that in a parametric form, it is easy to create many points on a given line with given properties; while in the usual form it is easy to tell if a given point is on a given line and also realize the position and orientation of the line in the plane.

Section 5.1 Learn how the points on a line in the plane (or higher dimensional space) can be associated to the values of a parameter t. This can be done by choosing two convenient points to correspond to the values $t = 0$ (an origin on the line) and $t = 1$ (a unit point on the line).

There are many such parametrizations for the same line and this flexibility is what makes the method useful.

Learn the parametric two point form as well as the corresponding compact form.

Learn the definition of direction numbers of the line. These give the slope of the line and lot more information at the same time.

Section 5.2 A chosen parametrization with a parameter t is used to deduce the position of a point in terms of the value of t at it.

Study the midpoint, division, exterior division and the distance formula in terms of t.

Section 5.3 Learn how a parametric form and the usual form are related and how to transform one to the other.

There are many crucial formulas in this section which should be memorized.

A special device to find a paralle or perpendicular line to a given line in usual form is described. It is more efficient than what you may have learned, since it does not require a slope intercept form.

Section 5.4 The power of the interplay between the two forms is illustrated by discussing various applications. These include intersecting two lines, finding perpendicular bisectors of given segments and constructing right angle triangles.

The important "duck principle" is introduced to show how to efficiently derive needed formulas.

Week 8

Read: Chapter 6, Chapter 7.1,7.2. In Chapter 6 we study the graphs of linear and quadratic functions. We deduce that the graphs are respectively lines or vertical parabolas. Also the vertex of the parabola (lowest or highest point depending on which way the parabola opens) is easily calculated.

Section 6.1 The graph of a linear function $L(x) = ax + b$ is a line with slope a. If $a > 0$, then the function steadily rises and it falls if $a < 0$. For $a = 0$, the function is constant and we get a horizontal line.

Section 6.2 The graph of a factored quadratic function $Q(x) = a(x - p)(x - q)$ is a parabola crossing the x-axis at $x = p$ and $x = q$.

For positive a, it opens up (or is concave up) and for negative a it opens down (or is concave down). The vertex of the parabola is at the average of p, q, i.e. at $x = \frac{p+q}{2}$.

This value of x gives an absolute extremum (minimum if $a > 0$ and maximum if $a < 0$).

If we expand $Q(x)$ as $Q(x) = ax^2 + bx + c$, then we have $\frac{p+q}{2} = -\frac{b}{2a}$.

Section 6.3 Next we prove that the graph of $Q(x) = ax^2 + bx + c$ is always a parabola with vertex at $x = -\frac{b}{2a}$ whether we can factor it or not. The sign of a decides the opening direction as above.

This calculation uses the important technique of "completing the square" learnt in Chapter 1.5 Example 6.

As a bonus, we derive the well known quadratic formula as well as a simple criterion if the parabola crosses the x-axis at two points ($b^2 - 4ac > 0$), or at just one point ($b^2 - 4ac = 0$) or at no point ($b^2 - 4ac < 0$).

Section 6.4 We apply the above theory to decide intervals on which a given quadratic function is increasing or decreasing. We develop an easy method of testing the values at special points.

Section 7.1,7.2 Chapter 7 discusses the nature of various types of functions (beyond linear and quadratic) and makes formal definition of the concept of a function (which we have used intutively so far).

We also define important related concepts of the domain, range and target.

We point out that not all functions have nice simple formulas and in real life situations there may be no hope of finding a formula.

We take up the problem of fitting a nice function to known data points next.

Week 9

This week is mainly spent in reviewing and memorizing important formulas and definitions. Review old homeworks and main calculation techniques.

The second exam takes place on Friday.

Week 10

Read: Chapter 7 and Chapter 8 along with Appendices 12.4, 12.5, 12.6. Be aware that the proofs may involve the concept of derivatives to be developed later. So, you may have to reread them later.

We finish up the study of Chapter 7 and enter into the study of circles in Chapter 8.

Section 7.3 Learn how to find a function of a desired type which passes through given data points. Sometimes this is not possible and we make a theory of finding a function which best fits the given data.

We especially learn how to fit linear or quadratic functions which are important in Statistics.

Section 7.3.1 In this section we define the important concepts of one to one and onto functions. A function having both these properties has an inverse. We study how to find the inverse function if possible.

Often, we can only prove the existence of an inverse function and prove its properties but not succeed in finding an explicit formula.

Some well known examples of this are the exponential and logarithmic function. These are treated in Appendix 12.4 and 12.6.

Section 8.1 Learn the basic equation of a circle and how to deduce the center and radius from a given expanded equation of a circle.

Section 8.2 Learn the important new fact that a circle also has parametric form like a line, but we need to use rational functions. Luckily, both the numerators and denominators are of degree at most 2.

Section 8.3 We study how the parametric form of a circle can give a complete solution in integers of the equation $x^2 + y^2 = z^2$. Such solutions are called Pythagorian triples and we get an easy method to generate them all.

We also discuss the fact that any general conic also has such a parametric form using rational functions, but do not pursue all the possible cases in this course.

Section 8.4 Armed with the usual and parametric forms of equations for a circle we develop many important formulas.

The following formulas and calculations are notable and deserve memorization:

Diameter form of a circle equation, line of intersection of two circles, circle through three given points, circle with a given center and tangent to a given line, distance between a point and a line, half planes defined by a line.

Some of the material from this section will spill into the next week.

Week 11

Read: Chapter 9.1-4. We finish of the discussion of the circle related topics from 8.4 and start with basic trigonometry.

Section 8.4 Remaining topics from previous week.

Section 9.1 We introduce another well known parameterization of the circle using trigonometric functions, namely $x = r\cos(t), y = r\sin(t)$. This is simpler to visualize, however, the functions are harder to compute, since they are defined by a power series or a geometric construction.

It does, in turn, give a complete definition of the angle and the trigonometric functions.

Learn the concept of the locator point $P(t)$ for a given angle t and the radian and degree measures of angles. Learn how to evaluate the trigonometric functions precisely for special angles. Be aware that for random angles only a calculator approximation is available.

Section 9.2 Memorize the definitions of all the six trigonometric functions and the first three fundamental identities.

Section 9.3 We now connect our new definitions of trigonometric functions with the old geometric definitions using right angle triangles.

Week 12

Read: Chapter 9 and Chapter 10.1.

Section 9.4 This section has several identities and it is crucial to memorize them and understand how they are used in the evaluations of trigonometric functions for related angles.

Also study the very important sine law and cosine law for analyzing parts of a triangle.

Section 9.5 Various practical applications of trigonometry are illustrated here.

Understand the technique of deriving a general formula for a given problem before using the known values of angles or distances. It often leads to cleaner calculations and the final answer solves a whole family of problems, not just the given one!

Section 9.6,9.7 These two sections are left for private study for a student who is inspired by complete proofs based on simple ideas. Once you understand the proofs you can use all the known identities with confidence. Moreover, you start the process of understanding how new results are proved.

Section 10.1 We start with applying two simple calculations to two familiar curves, namely parabola and a circle.

We shift origin to a chosen point on the respective curves and transform the equation in terms of the new coordinates. Then we keep only the linear terms of the resulting equations.

When the equation with just the linear terms is converted back to the original coordinates we get the tangent line to the original curve at the chosen point.

This linear equation is called the linear approximation of the original equation and the slope of the resulting line is the derivative of y with respect to x.

Thus, a simple algebraic manipulation lets us get into the topic of deriavtives which eventually becomes the heart of calculus. Be aware that we are handling only special functions (the so called algebraic functions) but these are the most used functions.

Week 13

This week is mainly spent in reviewing and memorizing important formulas and definitions. Review old homeworks and main calculation techniques.

The third exam takes place on Friday.

Week 14

Read: Chapter 10.2. This is a short week due to the thanksgiving period. There is only one lecture and one recitation.

Section 10.2 We discuss how to find the linear approximation to a function $f(x)$ by applying the process of Chapter 10.1 to the equation $y = f(x)$. We get the tangent line to the graph of this curve at a given point by the same process.

We then proceed to make convenient formulas to get the linear approximation and the tangent lines with very little work when $f(x)$ is a polynomial. This avoids the tedious process of shifting origin and collecting terms. In future, these formulas is all that would be needed (at least for polynomial functions).

Week 15

Read: Chapter 10.3,10.4, 10.5, 10.6. We continue the philosophy of Chapter 10.2 and learn to find the derivative (the slope of the intended tangent at the chosen point) with convenient formulas. Naturally, we get linear approximations and the equations of the tangent lines immediately.

Section 10.3 Memorize the five important formulas which make short work of derivatives of polynomial functions.

Section 10.4 Using the formulas from Chapter 10.4 and making an enhanced definition of the derivative of an algebraic function we find derivatives of many complicated functions (even without explicit formulas) by simple formulas. This is where the power of the algebraic method lies.

Section 10.5 We continue building more useful formulas (formula 6 and formula 7).

Section 10.6 Now we illustrate the use of the idea of linear approximation in estimating values of complicated functions.

Week 16

Read: Review all the topics studied so far.

This week is mainly spent in reviewing and memorizing important formulas and definitions. Review old homeworks and main calculation techniques.

The final exam takes place in the next week.

Week 17

This is the final exam week. You should review the whole course following the guidelines of what to study. **Best of luck!**