

Final Review with solved problems.

Ma 110 Fall 2008

Summary of topics to study.

1. Be sure to review the sample solved problems posted for the first three exams.
2. Be sure to review the first three exams and various homework problems for practice.
3. Be sure that the formulas you were supposed to have memorized, stay memorized!
4. Be sure to finish both the last homeworks with complete understanding. Sample problems on the same material (Chapter 10) are presented below.

Problems on Derivatives.

These problems did not exist in the final exam of Fall 2007. So, this is the only sample of exam problems based on this material.

1. Derivative by shifting and linearizing.

- (a) If $f(x) = 2 + 3x - 5x^2$, then the equation of the tangent line to its graph at $x = -1$ is $y = \underline{\hspace{2cm}} x + \underline{\hspace{2cm}}$.

Answer. We can take $a = -1$ and then $b = f(a) = 2 + 3(-1) - 5(-1)^2 = -6$.

As usual, we substitute $x = u + a = u - 1$, $y = v + b = v - 6$. Then the equation $y - f(x) = 0$ transforms as:

$$\begin{aligned}y - 2 - 3x + 5x^2 &= 0 \\(v - 6) - 2 - 3(u - 1) + 5(u - 1)^2 &= 0 \\v + (-6 - 2 + 3 + 5) + (-3 + 5(-2))u + 5u^2 &= 0 \\v + (0) + (-13)u + 5u^2 &= 0 \\v - 13u + \text{higher degree terms} &= 0.\end{aligned}$$

Thus, the tangent line equation is:

$$v - 13u = 0 \text{ or } (y + 6) - 13(x + 1) = 0 \text{ which becomes } y = 13x + 7.$$

- (b) The equation of the tangent line to the ellipse $x^2 + 5y^2 + x + y = 12$ at the point $x = 2, y = 1$ is: $y = \underline{\hspace{2cm}} x + \underline{\hspace{2cm}}$.

Answer.

Here we make the substitution $x = u + 2, y = v + 1$ and simplify the equation in steps.

$$(u + 2)^2 + 5(v + 1)^2 + (u + 2) + (v + 1) = 12.$$

$$(u^2 + 4u + 4) + (5v^2 + 10v + 5) + (u + 2) + (v + 1) = 12.$$

Throwing away higher degree terms, we get the tangent line equation:

$$4u + 4 + 10v + 5 + u + 2 + v + 1 = 12 \text{ or } 11v + 5u = 0.$$

Going back to x, y we get:

$$11(y - 1) + 5(x - 2) = 0 \text{ or } 11y = -5x + 21 \text{ or } y = -\frac{5}{11}x + \frac{21}{11}.$$

2. **Derivative formulas.** It is often possible to see the answer by inspection alone, but you need to justify your work by indicating what you used during the exam. This is illustrated below.

- (a) Let $f(x) = x^2 + 5x - 3, g(x) = x^3 + x^2 - 1$.

Calculate the indicated quantities.

Comment: First note that $f'(x) = 2x + 5$ and $g'(x) = 3x^2 + 2x$ by the combined polynomial formula.

- i. $(2f + 3g)'(x)$ **Answer.**

By the sum and the constant multiplier rules:

$$(2f + 3g)'(x) = (2f)'(x) + (3g)'(x) = 2f'(x) + 3g'(x)$$

and this evaluates as:

$$2(2x + 5) + 3(3x^2 + 2x) = 9x^2 + 10x + 10.$$

- ii. Find the equation of the tangent line to the graph of $y = f(x)g(x)$ at $x = 1$.

Answer.

First, by product rule:

$$D_x(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

and this evaluates as:

$$(x^2 + 5x - 3)(3x^2 + 2x) + (2x + 5)(x^3 + x^2 - 1).$$

The value of this derivative at $x = 1$ is:

$$(1 + 5 - 3)(3 + 2) + (2 + 5)(1 + 1 - 1) = 15 + 7 = 22.$$

The value of $f(x)g(x)$ at $x = 1$ is $(1 + 5 - 3)(1 + 1 - 1) = 3$.
(Note that we were not eager to simplify the expression before substitution. Sometimes, this is easier!)

Thus the tangent line passes through $(1, 3)$ and has slope 22.

Thus its equation is: $(y - 3) = 22(x - 1)$ or $y = 22x - 19$.

- iii. Find the derivative of $(x^3 + x)^7$ at $x = -1$.

Answer.

By the enhanced power rule

$$D_x(x^3 + x)^7 = 7(x^3 + x)^6(D_x(x^3 + x)) = 7(x^3 + x)^6(3x^2 + 1)$$

where we have used the combined polynomial rules for the last simplification.

Evaluation at $x = -1$ gives $7(-1 - 1)^6(3 + 1) = 7(64)(4) = 1792$.

- iv. Find the derivative of $\frac{x + 1}{x^2 + 3x + 7}$.

Answer. Set $y = x + 1$, then $D_x(y) = 1$. Set $z = (x^2 + 3x + 7)$, then $D_x(z) = 2x + 3$.

The quotient rule gives:

$$\frac{(1)(x^2 + 3x + 7) - (x + 1)(2x + 3)}{(x^2 + 3x + 7)^2} = \frac{(x^2 + 3x + 7) - (2x^2 + 5x + 3)}{(x^2 + 3x + 7)^2}$$

and this gives the final answer:

$$\frac{-x^2 - 2x + 4}{(x^2 + 3x + 7)^2}.$$

The last simplification is not essential, but the practice is necessary for future success in such problems where we may have to take another derivative. So, do practice the extra steps.

(b) **Derivative by inspection.**

Suppose the graph of $y = f(x)$ has tangent line $y = 6x + 7$ at $(1, 13)$. Also, the graph of $y = g(x)$ has tangent line $y = -3x + 16$ at the same point.

Calculate the derivative of the function $h(x) = (f(x))^2 + 3g(x)$ for $x = 1$. use it to find the equation of the tangent line to the graph of $y = h(x)$ at $x = 1$.

Answer.

By the power, product and sum rules, we know that:

$$h'(x) = 2f(x)f'(x) + 3g'(x) \text{ and hence } h'(1) = 2f(1)f'(1) + 3g'(1).$$

From the given tangent lines, we deduce $f'(1) = 6$, $g'(1) = -3$ and we know that $f(1) = g(1) = 13$.

So, $h'(1) = 2(13)(6) + 3(-3) = 156 - 9 = 147$. Also, $h(1) = f(1)^2 + 3g(1) = 13^2 + 3(13) = 208$.

So, the desired tangent line is given by $y - h(1) = h'(1)(x - 1)$ or $(y - 208) = 147(x - 1)$.