The Cauchy Property

1) They are *not* real numbers and do *not* necessarily obey the rules of arithmetic for real numbers.
2) We often act as if they do.
3) We need guidelines.

Add $\infty$ and $-\infty$ to $\mathbb{R}$ and extend the ordering by $-\infty < a < +\infty$
for every real number $a \in \mathbb{R} \cup \{+\infty, -\infty\}$.

If $a \in \mathbb{R}$ then we define the following:

1) $a + \infty = +\infty$
2) $a - \infty = -\infty$
3) If $a > 0$, then $a \times \infty = \infty$ and $a \times -\infty = -\infty$
4) If $a < 0$, then $a \times \infty = -\infty$ and $a \times -\infty = +\infty$

We may adopt the following conventions:

$a/\infty = 0$ and $a/(-\infty) = 0$
Limits of Sequences

Limit of \( \{a_n\} \) exists IFF we can compute \( L \).

Will this always work?

Can we always find the limit?

Do we have to be able to find the limit as a number?

Theorem

**Theorem (last lecture):** Every convergent sequence is bounded.

Is the converse true?

Is it true that every bounded sequence converges?

Find a proof or a counterexample.

Definitions

A sequence \( \{a_n\} \) is **increasing** if \( a_n \leq a_{n+1} \) for every \( n \).

A sequence \( \{a_n\} \) is **decreasing** if \( a_n \geq a_{n+1} \) for every \( n \).

A sequence is **monotone (monotonic)** if it is either increasing or decreasing.
Examples
1) Find an example of an increasing sequence.
2) Find an example of a decreasing sequence.
3) Find an example of a sequence that is not monotonic.

Increasing Sequences

Decreasing Sequences
Non-monotonic Sequences

Monotone Convergence Theorem

Theorem: Every bounded monotonic sequence converges.

Proof:
Let \( \{a_n\} \) be a bounded increasing sequence and let \( S = \{a_n \mid n \in \mathbb{N}\} \). Since the sequence is bounded, \( a_n < M \) for some real number \( M \) and for all \( n \).
Therefore \( S \) is bounded and has a least upper bound. Let \( u = \text{lub} \ S \) and let \( \epsilon > 0 \).

Theorem

Proof:
Since \( u = \text{lub} \ S \) and \( \epsilon > 0 \), \( u - \epsilon \) is not an upper bound for \( S \). Thus there is an integer \( K \) so that \( a_K > u - \epsilon \). Since \( \{a_n\} \) is increasing then for all \( n > K \), \( a_n \geq a_K \) and for all \( n > K \)
\( u - \epsilon < a_n \leq u \).
Thus, \( |a_n - u| < \epsilon \) for all \( n > K \) and \( \lim a_n = u = \text{lub} \ S \).
Consequences

1) The decimal representation of a real number converges.
\[ m < m.d_1d_2d_3\ldots = m + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \leq m + 1 \]

Let \( a_n = m.d_1d_2d_3\ldots \). Then \( a_n \leq a_{n+1} \), so \( \{a_n\} \) is increasing.

2) Let \( a_0 = 1 \) and \( a_{n+1} = \frac{1}{1 + a_n} \)

\[
\begin{align*}
\text{Let} & \quad a_0 = 1 \quad \text{and} \quad a_{n+1} = 1 + \sqrt{a_n} \\
\text{Does it converge? Is it monotone?} & \\
a_0 & = 1 \\
a_1 & = 1 + \sqrt{a_0} = 2 \\
a_2 & = 1 + \sqrt{a_1} = 1 + \sqrt{2} \approx 2.4142 \\
a_3 & = 1 + \sqrt{a_2} = 1 + \sqrt{2.4142} \approx 2.55377 \\
\text{Prove it is increasing by induction on } n. \end{align*}
\]

Consequences

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\text{Prove it is increasing by induction on } n. \end{align*}
\]

Consequences

2) Let \( a_0 = 1 \) and \( a_{n+1} = 1 + \sqrt{a_n} \).
Converges by Monotone Convergence Theorem. To what does it converge?
Assume: \( \lim_{n \to \infty} a_n = L \)
\[
\begin{align*}
a_{n+1} & = 1 + \sqrt{a_n} \\
\lim_{n \to \infty} a_{n+1} & = 1 + \lim_{n \to \infty} \sqrt{a_n} \\
L & = 1 + \sqrt{L} \\
(L - 1)^2 & = L \quad \text{so} \quad L^2 - 3L + 1 = 0 \\
L & = \frac{3 \pm \sqrt{(9 - 4)}}{2} = \frac{3 \pm \sqrt{5}}{2} \\
\text{Which one is it? It cannot be both. Why?} \end{align*}
\]
Theorem

**Theorem**: Let \( \{a_n\} \) be a sequence of real numbers.

(i) If \( \{a_n\} \) is an unbounded monotonically increasing sequence, then \( \lim a_n = +\infty \).

(ii) If \( \{a_n\} \) is an unbounded monotonically decreasing sequence, then \( \lim a_n = -\infty \).

\[ 10/29/2008 \]

Theorem

**Theorem**: Suppose that \( \{a_n\} \) is a monotone increasing sequence and \( \{b_n\} \) is a monotone decreasing sequence such that 

\[ a_n \leq b_n \text{ for all } n = 0, 1, 2, \ldots \]

and

\[ \{a_n - b_n\} \to 0 \]

Then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \).

\[ 29-Oct-2008 \]

Theorem

**Theorem**: Every sequence contains a monotone subsequence.

Proof: Let \( \{a_n\} \) be a sequence. We say that a term \( a_n \) is dominating if \( a_n > a_m \) for all \( m > n \).

Claim: Every sequence contains an infinite number or a finite number of dominating terms. (Note: finite could be 0.)
Theorem

Proof (continued):
(i) Assume \( \{a_n\} \) has an infinite number of dominating terms. Call these \( a_{n_0}, a_{n_1}, a_{n_2}, \ldots \) where \( n_0 < n_1 < n_2 < \ldots \). By definition
\[
a_{n_0} > a_{n_1} > a_{n_2} > \ldots
\]
which is the monotone subsequence.

(ii) Assume \( \{a_n\} \) has a finite number of dominating terms. Thus, there is an \( m \) so that for every \( n > m \), \( a_n \) is not dominating. That means that for each \( n > m \) there exists a \( k > n \) so that \( a_n \leq a_k \). Let \( n_0 = m \). By the above there is a \( n_1 > n_0 \) so that \( a_{n_0} \leq a_{n_1} \). Since \( n_1 > n_0 \) then there is \( n_2 > n_1 \) so that \( a_{n_1} \leq a_{n_2} \). This gives
\[
a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \ldots
\]
which is the required monotone subsequence.

Bolzano-Weierstrauss Theorem

**Theorem:** Every bounded sequence has a convergent subsequence.
The Cauchy Property

**Definition 1:** A sequence \( \{a_n\} \) is said to have the Cauchy property if for every \( \varepsilon > 0 \) there is an index \( K \) so that
\[
| a_{n+m} - a_n | < \varepsilon
\]
for all \( n \geq K \) and \( m = 1, 2, 3, \ldots \)
[Note: equivalent statement –
\( \{a_{n+m}\}_{m=0}^{\infty} \subset (a_n - \varepsilon, a_n + \varepsilon) \) for all \( n \geq K \).]

The Cauchy Property

**Definition 2:** A sequence \( \{a_n\} \) is said to have the Cauchy property if for every \( \varepsilon > 0 \) there is an index \( K \) so that if \( n, m > K \) then
\[
| a_m - a_n | < \varepsilon.
\]

Definitions

Let \( \{a_n\} \) be bounded – convergent or not, it does not matter.
Limiting behavior of \( \{a_n\} \) depends only on the **tails** of the sequence, \( \{a_n \mid n > N\} \).

Let \( u_N = \text{glb}\{a_n \mid n > N\} \)
Let \( v_N = \text{lub}\{a_n \mid n > N\} \)
FACT: If \( \lim a_n \) exists, then it lies in \([u_N, v_N]\).
**Definitions**

As $N$ increases, the sets $\{a_n \mid n > N\}$ get smaller. Thus,

$$u_1 \leq u_2 \leq u_3 \leq \ldots \quad \text{and} \quad v_1 \geq v_2 \geq v_3 \geq \ldots$$

Let

$$u = \lim_{N \to \infty} u_N \quad \text{and} \quad v = \lim_{N \to \infty} v_N$$

Both exist – Why?

Claim: $u \leq v$

**Definitions**

If $\lim_{n \to \infty} a_n$ exists, then $u_N \leq \lim a_n \leq v_N$

so $u \leq \lim a_n \leq v$.

$u$ and $v$ are useful whether $\lim a_n$ exists or not.

Definition:

$$u = \limsup a_n = \lim(\lub \{a_n \mid n > N\})$$

and

$$v = \liminf a_n = \lim(\glb \{a_n \mid n > N\})$$

**lim inf and lim sup**

Note: Do not require that $\{a_n\}$ be bounded.

Precautions and Conventions.

1) If $\{a_n\}$ is not bounded above, lub $\{a_n\} = +\infty$ and we define $\limsup a_n = +\infty$

2) If $\{a_n\}$ is not bounded below, glb $\{a_n\} = -\infty$ and we define $\liminf a_n = -\infty$. 
lim inf and lim sup

Is it true that \( \limsup \{a_n\} = \text{lub} \{a_n\} \)?
Not necessarily, because while it is true that \( \limsup \{a_n\} \leq \text{lub} \{a_n\} \),
some of the values \( a_n \) may be much larger than \( \limsup a_n \).

Note that \( \limsup a_n \) is the largest value that \( \text{infinitely many} \ a_n \)'s can get close to.

**Theorem**: Let \( \{a_n\} \) be a sequence of real numbers.

(i) If \( \lim a_n \) is defined [as a real number, \( +\infty \) or \( -\infty \)], then \( \liminf a_n = \lim a_n = \limsup a_n \).

(ii) If \( \liminf a_n = \limsup a_n \), then \( \lim a_n \) is defined and \( \lim a_n = \liminf a_n = \limsup a_n \).

**Proof**

Let \( u_N = \text{glb} \{a_n \mid n > N\} \), \( v_N = \text{lub} \{a_n \mid n > N\} \),
\( u = \lim u_N = \liminf a_n \) and
\( v = \lim v_N = \limsup a_n \).

(i) Suppose \( \lim a_n = +\infty \). Let \( M > 0 \). There is \( N \in \mathbb{N} \) so that if \( n > N \) then \( a_n > M \). Then
\( u_N = \text{glb} \{a_n \mid n > N\} \geq M \).
So if \( m > N \) then \( u_m \geq M \).
Therefore \( \lim u_N = \liminf a_n = +\infty \). Likewise,
\( \limsup a_n = +\infty \).
Do the case that \( \lim a_n = -\infty \) similarly.
Proof

Suppose that \( \lim a_n = L \in \mathbb{R} \). Let \( \varepsilon > 0 \). There is \( N \in \mathbb{N} \) so that \( |a_n - L| < \varepsilon \) for \( n > N \).

\( a_n < L + \varepsilon \) for \( n > N \).

Thus \( v_N = \text{lub}\{a_n \mid n > N\} \leq L + \varepsilon \).

If \( m > N \) then \( v_m \leq L + \varepsilon \) for all \( \varepsilon > 0 \).

Thus \( \limsup a_n \leq L = \lim a_n \).

Similarly, show that \( \liminf a_n \leq \limsup a_n \).

Since \( \liminf a_n \leq \limsup a_n \), we have \( \liminf a_n = \lim a_n = \limsup a_n \).

Proof

(ii) If \( \liminf a_n = \limsup a_n = \pm \infty \) easy to show that \( \lim a_n = \pm \infty \).

Suppose that \( \liminf a_n = \limsup a_n = L \). We need to show that \( \lim a_n = L \).

Let \( \varepsilon > 0 \). Since \( L = \lim v_N \) there is an \( N_0 \in \mathbb{N} \) so that

\[ |L - \text{lub}\{a_n \mid n > N_0\}| < \varepsilon. \]

Thus, \( \text{lub}\{a_n \mid n > N_0\} < L + \varepsilon \) and

\[ a_n < L + \varepsilon \text{ for all } n > N_0. \]

These imply \( L - \varepsilon < a_n < L + \varepsilon \) for \( n > \max\{N_0, N_1\} \).

Equivalently, \( |a_n - L| < \varepsilon \) for \( n > \max\{N_0, N_1\} \).

This proves that \( \lim a_n = L \).

Proof

Similarly, since \( L = \lim u_N \) there is \( N_1 \in \mathbb{N} \) so that

\[ |L - \text{glb}\{a_n \mid n > N_1\}| < \varepsilon. \]

Thus, \( \text{glb}\{a_n \mid n > N_1\} > L - \varepsilon \) and

\[ a_n > L - \varepsilon \text{ for all } n > N_1. \]

These imply \( L - \varepsilon < a_n < L + \varepsilon \) for \( n > \max\{N_0, N_1\} \).

Equivalently, \( |a_n - L| < \varepsilon \) for \( n > \max\{N_0, N_1\} \).

This proves that \( \lim a_n = L \).
Theorems

Lemma:
Convergent sequences have the Cauchy property.

Proof:
Suppose that \( \lim a_n = L \).
\[
| a_n - a_m | = | a_n - L + L - a_m | \leq | a_n - L | + | a_m - L |
\]
Let \( \varepsilon > 0 \), there is an integer \( N \) so that if \( k > N \),
\[
| a_k - L | < \varepsilon/2. \text{ If } m, n > N \text{ then}
\]
\[
| a_n - a_m | \leq | a_n - L | + | a_m - L | < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
Thus, \( \{a_n\} \) has the Cauchy property.

Theorem:
A sequence is a convergent sequence if and only if it has the Cauchy property.

Proof: The previous lemma proves half of this.
Show: any sequence with the Cauchy property must converge. Let \( \{a_n\} \) have the Cauchy property. We know it is bounded by the previous lemma.
Show: \( \lim \inf a_n = \lim \sup a_n \).
Proof

Let $\varepsilon > 0$. Since $\{a_n\}$ has the Cauchy property, there is an $N \in \mathbb{N}$ so that if $m, n > N$ then $|a_n - a_m| < \varepsilon$. In particular, $a_n < a_m + \varepsilon$ for all $m, n > N$. This shows that $a_m + \varepsilon$ is an upper bound for $\{a_n \mid n > N\}$. Thus $v_N = \text{lub}\{a_n \mid n > N\} \leq a_m + \varepsilon$ for $m > N$. This shows that $v_N - \varepsilon$ is a lower bound for $\{a_m \mid m > N\}$, so $v_N - \varepsilon \leq \text{glb}\{a_m \mid m > N\} = u_N$.

Therefore

$$\limsup a_n \leq v_N \leq u_N + \varepsilon \leq \liminf a_n + \varepsilon$$

Since this holds for all $\varepsilon > 0$, we have that

$$\limsup a_n \leq \liminf a_n$$

This is enough to give us that the two quantities are equal.

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and } a_{n+1} = a_n + \frac{1}{a_n}$$
Problems
Compute the limit if it exists:

\[ a_0 = 1 \text{ and } a_{n+1} = 3 - \frac{1}{a_n} \]

Problems
Compute the limit if it exists:

\[ a_0 = 0 \text{ and } a_{n+1} = \frac{a_n + 1}{a_n + 2} \]

Problems
Compute the limit if it exists:

\[ a_0 = 1 \text{ and } a_{n+1} = \frac{a_n + 1}{a_n + 2} \]
Problems

Compute the limit if it exists:

\[ a_0 = 0 \text{ and } \]
\[ a_{n+1} = a_n^2 + \frac{1}{4} \]