MATH 6101
Fall 2008
Newton and Differential Equations
A Differential Equation

What is a differential equation?

A differential equation is an equation relating the quantities $x$, $y$ and $y'$ and possibly higher derivatives of $y$.

Examples: 

$y' = x + y$

$y'' - y = 0$

$(1 + x^2)y'' + 2xy' + 4x^2y = 0$
Differential Equations

How are differential equations used?

Newton’s Second Law:

\[ F = ma = \frac{d(mv)}{dt} = m \frac{d^2s}{dt^2} \]

Radioactive decay:

\[ \frac{dP}{dt} = -kP \]
Differential Equations

Newton’s Law of Cooling:

\[
\frac{dQ}{dt} = h \cdot A(T_0 - T_{env})
\]

The wave equation:

\[
\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 u
\]

The heat equation:

\[
\frac{\partial u}{\partial t} - k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0
\]
Consider the following simple differential equation:

\[ \frac{dP}{dt} = k \cdot P \]

Our “modern process” is *Separation of Variables* introduced by l’Hospital in 1750.
Separation of Variables

\[ \frac{dP}{dt} = k \cdot P \]

\[ \frac{dP}{P} = k \cdot dt \]

\[ \int \frac{dP}{P} = \int k \cdot dt = k \int dt \]

\[ \ln(P) + C_1 = kt + C_2 \]

\[ \ln(P) = kt + C \]

\[ P(t) = e^{kt + C} = Ae^{kt} \]
Newton’s Method of Series

\[ \frac{dP}{dt} = k \cdot P \]

Now, assume that we can express \( P \) as a function of \( t \) by:

\[ P(t) = \sum_{n=0}^{\infty} a_n t^n \]

Then

\[ \frac{dP}{dt} = \sum_{n=1}^{\infty} n a_n t^{n-1} \]

and our first equation gives...
Newton’s Method of Series

\[ \frac{dP}{dt} = k \cdot P \]

\[ \sum_{n=1}^{\infty} n a_n t^{n-1} = k \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} k a_n t^n \]

\[ a_1 t^0 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = k a_0 t^0 + k a_1 t + k a_2 t^2 + k a_3 t^3 + \cdots \]

Setting corresponding powers of \( t \) equal, gives the following
Newton’s Method of Series

\[ a_1 = k a_0 \]

\[ 2a_2 = ka_1 \Rightarrow a_2 = \frac{k}{2} a_1 = \frac{k^2}{2} a_0 \]

\[ 3a_3 = ka_2 \Rightarrow a_3 = \frac{k}{3} a_2 = \frac{k^3}{2 \cdot 3} a_0 \]

\[ 4a_4 = ka_3 \Rightarrow a_4 = \frac{k}{4} a_3 = \frac{k^4}{2 \cdot 3 \cdot 4} a_0 \]

\[ \vdots \]

\[ na_n = ka_{n-1} \Rightarrow a_n = \frac{k}{n} a_{n-1} = \frac{k^n}{n!} a_0 \]
Newton’s Method of Series

\[ P(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_0 \frac{k^n}{n!} t^n = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \]

We will show later that

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

Therefore,

\[ P(t) = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = a_0 e^{kt} \]
Newton’s Method of Series

Example 2: Find a solution to the following equation

\[ y'' + y = 0 \]

Let

\[ y = \sum_{n=0}^{\infty} a_n x^n \]

Then

\[ y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \]
Newton’s Method of Series

\[ y'' + y = 0 \]
\[ y'' = -y \]

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} a_n x^n
\]

\[ 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_4 x^3 + \cdots = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \cdots \]

This gives us that

\[ 2a_2 = -a_0 \quad 4 \cdot 3a_4 = -a_2 \quad 6 \cdot 5a_6 = -a_4 \quad 8 \cdot 7a_8 = -a_6 \]
\[ 3 \cdot 2a_3 = -a_1 \quad 5 \cdot 4a_5 = -a_3 \quad 7 \cdot 6a_7 = -a_5 \quad 9 \cdot 8a_9 = -a_7 \]
Newton’s Method of Series

\[ 2a_2 = -a_0 \Rightarrow a_2 = -\frac{1}{2!}a_0 \]

\[ 3 \cdot 2a_3 = -a_1 \Rightarrow a_3 = -\frac{1}{3!}a_1 \]

\[ 4 \cdot 3a_4 = -a_2 \Rightarrow a_4 = -\frac{1}{4!}a_2 = -\frac{1}{2 \cdot 3 \cdot 4} \cdot a_0 = -\frac{1}{4!}a_0 \]

\[ 5 \cdot 4a_5 = -a_3 \Rightarrow a_5 = -\frac{1}{5!}a_3 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \cdot a_1 = -\frac{1}{5!}a_1 \]

\[ 6 \cdot 5a_6 = -a_4 \Rightarrow a_6 = -\frac{1}{6!}a_4 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot a_0 = -\frac{1}{6!}a_0 \]

\[ 7 \cdot 6a_7 = -a_5 \Rightarrow a_7 = -\frac{1}{7!}a_5 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot a_1 = -\frac{1}{7!}a_1 \]
Newton’s Method of Series

\[ y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \]

\[ = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \frac{1}{6!} a_0 x^6 - \frac{1}{7!} a_1 x^7 + \cdots \]

\[ = a_0 \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots \right) + a_1 \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \right) \]

Let

\[ c(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots \]

and

\[ s(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \]
Newton’s Method of Series

This gives us that

\[ y = a_0 c(x) + a_1 s(x) \]

Note that:

\[
[c(x)]^2 + [s(x)]^2 = \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots \right)^2 + \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \ldots \right)^2
\]

\[
= 1 - \frac{2}{2!} x^2 + \frac{1}{2!2!} x^4 + \frac{2}{4!} x^4 - \frac{2}{2!4!} x^6 - \frac{2}{6!} x^6 + x^2 - \frac{2}{3!} x^4 + \frac{1}{3!3!} x^6 + \frac{2}{5!} x^6 + \ldots
\]

\[
= 1 + (1 - 1)x^2 + \left(\frac{1}{2!2!} + \frac{2}{4!} - \frac{2}{3!}\right) x^4 + \left(-\frac{2}{2!4!} - \frac{2}{6!} + \frac{1}{3!3!} + \frac{2}{5!}\right) x^6 + \ldots
\]

\[
= 1 + 0x^2 + \left(\frac{1}{4} + \frac{1}{12} - \frac{1}{3}\right) x^4 + \left(-\frac{1}{24} - \frac{1}{360} + \frac{1}{36} + \frac{1}{60}\right) x^6 + \ldots
\]

\[
= 1 + 0x^2 + \left(\frac{4}{12} - \frac{4}{12}\right) x^4 + \left(-\frac{16}{360} + \frac{16}{360}\right) x^6 + \ldots
\]

\[
= 1
\]
Newton’s Method of Series

Also note that:

\[ c(0) = 1 \]
\[ s(0) = 0 \]

Further note that:

\[ \frac{d}{dx} \cos(x) = -\cos(x) \]
\[ \frac{d}{dx} \sin(x) = -\sin(x) \]

We can show that:

\[ c(x) = \cos(x) \]
\[ s(x) = \sin(x) \]
Your Turn

Solve by Newton’s method (i.e., find the first 5 non-zero terms):

1) \[ y' = y + \frac{1}{1 - x} \]

2) \[ y' = 1 + xy \]
Solutions

\[ y' = y + \frac{1}{1-x} \]
\[ y(x) = a_o + (1 + a_o)x + \frac{1}{2}a_o x^2 + \frac{1}{3} \left(1 + \frac{1}{2}a_o\right)x^3 + \frac{1}{6} \left(\frac{1}{4}a_o - 1\right)x^4 + \frac{1}{6} \left(1 + \frac{1}{20}a_o\right)x^5 + \cdots \]

\[ y' = 1 + xy \]
\[ y(x) = a_o + x + \frac{1}{2}a_o x^2 + \frac{1}{3} x^3 + \frac{1}{8}a_o x^4 + \frac{1}{15} x^5 + \cdots \]
Solving Algebraic Equations

How can we solve an equation of the following form?

\[ y^2 + 3y - 4 - x^2y + 2x = 0 \]

For this technique we do the same: assume that \( y \) has a power series expansion in terms of \( x \).

\[ y = \sum_{n=0}^{\infty} a_n x^n \]

Where does this lead us?
How do we cube out a series? This seems to be the only sticking point to proceeding with this process.

We will make a couple of definitions and assumptions.
Solving Algebraic Equations

\[ y = \sum_{n=0}^{\infty} a_n x^n \]

Given the above series define the tail series to be

\[ t_n = \sum_{k=n}^{\infty} a_n x^n = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots \]

We then have that

\[ t_0 = y \quad \text{and} \quad t_n = a_n x^n + t_{n+1} \]
Solving Algebraic Equations

Given an algebraic equation we will write it as

\[ F(x,y) = 0 \]

And we will begin our procedure by setting

\[ F_0(x,y) = F(x,y) \]

The \( k \)th iteration of our process consists of three steps

1) Extract the \( x^k \) level equation from \( F_k(x,t_k) = 0 \). That is find the coefficients of \( x^k \) and set them equal to 0.
2) Solve the \( x^k \) level equation for \( a_k \).
3) Substitute \( t_k = a_k x^k + t_{k+1} \) into \( F_k(x,t_k) = 0 \) and use the Binomial Theorem to obtain the new equation

\[ F_{k+1}(x,t_{k+1}) \]
Solving Algebraic Equations

Example:

\[ F(x,y) = y^2 + 3y - 4 - x^2y + 2x = 0 \]

\[ F_o(x,t_o) : \quad t_o^2 + 3t_o - 4 - x^2t_o + 2x = 0 \]

Step 1: Since \( t_o = a_o + a_1x + a_2x^2 + \ldots \) and the \( x^0 \) level equation ignores all non-constant terms in \( F_o(x,t_o) \):

\( x^0 \) level:

\[ a_o^2 + 3a_o - 4 = 0 \quad \Rightarrow \quad (a_o + 4)(a_o - 1) = 0 \quad \Rightarrow \quad a_o = -4 \text{ or } a_o = 1 \]

This will give us two solutions and two equations
Solving Algebraic Equations

Solution 1: \( a_0 = 1 \)

Now, replace \( t_0 \) with \( 1 + t_1 \) to get:

\[
(1 + t_1)^2 + 3(1 + t_1) - 4 - x^2(1 + t_1) + 2x = 0
\]

\[
1 + 2t_1 + t_1^2 + 3 + 3t_1 - 4 - x^2 - t_1x^2 + 2x = 0
\]

\[
5t_1 + t_1^2 - x^2 - t_1x^2 + 2x = 0
\]

Ignore all terms with degree > 1.

\[
5a_1 + 2 = 0
\]

\[
a_1 = -\frac{2}{5}
\]
Solving Algebraic Equations

\[ F_1(x, t_1) = 5t_1 + t_1^2 - x^2 - t_1 x^2 + 2x = 0 \]

Now, replace \( t_1 \) with \( -\frac{2}{5}x + t_2 \)

\[ 5t_1 + t_1^2 - x^2 - t_1 x^2 + 2x = 0 \]

\[ 5\left(-\frac{2}{5}x + t_2\right) + \left(-\frac{2}{5}x + t_2\right)^2 - x^2 - \left(-\frac{2}{5}x + t_2\right)x^2 + 2x = 0 \]

\[ 5t_2 + t_2^2 - \frac{4}{5}xt_2 - x^2t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0 \]

Ignore all terms with degree > 2.

\[ 5a_2 - \frac{21}{25} = 0 \]

\[ a_2 = \frac{21}{125} \]
Solving Algebraic Equations

\[ F_2(x,t_2) = 5t_2 + t_2^2 - \frac{4}{5}xt_2 - x^2t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0 \]

Now, replace \( t_2 \) with \( \frac{21}{125}x^2 + t_3 \)

\[ 5t_3 + t_3^2 - \frac{4}{5}xt_3 - \frac{83}{125}x^2t_3 + \frac{166}{625}x^3 - \frac{2184}{15625}x^4 = 0 \]

Ignore all terms with degree > 3.

\[ 5a_3 + \frac{166}{625} = 0 \]

\[ a_3 = -\frac{166}{3125} \]
Solving Algebraic Equations

So, we find a series expansion for $y$ of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{2}{5} x + \frac{21}{125} x^2 - \frac{166}{3125} x^3 + \cdots$$
Your Turn

1) Find the solution corresponding to $a_0 = -4$

2) Find the first four terms of the infinite series expansion of $y$ if $y^3 + xy = 1$. 
Your Turn

1) We can solve the original equation for $y$ and then find the series expansion for that. We should get the same answer.

\[ y^2 + 3y - 4 - x^2y + 2x = 0 \]
\[ y^2 + (3 - x^2)y + (2x - 4) = 0 \]

\[
\begin{align*}
y & = \frac{(x^2 - 3) \pm \sqrt{(3 - x^2)^2 - 4(2x - 4)}}{2} \\
y & = \frac{(x^2 - 3) \pm \sqrt{x^4 - 6x^2 - 8x + 25}}{2}
\end{align*}
\]
Your Turn

\[ y_1 = \frac{x^2 - 3 + \sqrt{x^4 - 6x^2 - 8x + 25}}{2} \]

\[ = 1 - \frac{2}{5}x + \frac{21}{125}x^2 - \frac{166}{3125}x^3 + \frac{304}{15625}x^4 + \cdots \]

\[ y_2 = \frac{x^2 - 3 - \sqrt{x^4 - 6x^2 - 8x + 25}}{2} \]

\[ = -4 + \frac{2}{5}x + \frac{104}{125}x^2 + \frac{166}{3125}x^3 - \frac{304}{15625}x^4 + \cdots \]

This is what we should have found with our other process.
Your Turn

2) Find the first four terms of the infinite series expansion of $y$ if $y^3 + xy = 1$.

\[ y^3 + xy - 1 = 0 \]

\[
y = \frac{1}{6} \sqrt[3]{108 + 12\sqrt{12}x^3 + 81} - \frac{2x}{3\sqrt[3]{108 + 12\sqrt{12}x^3 + 81}} \]

\[
y = 1 - \frac{1}{3}x + \frac{1}{81}x^3 + \frac{1}{243}x^4 + \ldots
\]
Problem 2

\[ F(x,y) = y^3 + xy - 1 = 0 \]

Stage 0:  \[ F_0(x,t_0) = t_0^3 + xt_0 - 1 = 0 \]

\[ a_0^3 - 1 = 0 \]

\[ a_0 = 1 \]

Stage 1: In \[ F_0(x,t_0) \] replace \[ t_0 = 1 + t_1 \]

\[ F_1(x,t_1) = (1 + t_1)^3 + x(1 + t_1) - 1 = 0 \]

\[ F_1(x,t_1) = 3t_1 + 3t_1^2 + t_1^3 + xt_1 + x = 0 \]

\[ 3a_1 + 1 = 0 \]

\[ a_1 = -\frac{1}{3} \]
Problem 2

Stage 2: In $F_1(x,t_1)$ replace $t_1 = -\frac{1}{3}x + t_2$

\[
F_1(x,t_1) = 3t_1 + 3t_1^2 + t_1^3 + xt_1 + x = 0
\]

\[
F_2(x,t_2) = 3\left(-\frac{1}{3}x + t_2\right) + 3\left(-\frac{1}{3}x + t_2\right)^2 \pm \left(1 + \frac{1}{3}x + t_2\right)^3 + x\left(-\frac{1}{3}x + t_2\right) + x = 0
\]

\[
= 3t_2 + 3t_2^2 + t_2^3 - xt_2 + \frac{1}{3}x^2t_2 - xt_2^2 - \frac{1}{27}x^3 = 0
\]

$3a_2 = 0$

$a_2 = 0$
Problem 2

Stage 3: In $F_2(x,t_2)$ replace $t_2 = 0x^2 + t_3 = t_3$

$$F_2(x,t_2) = 3t_2 + 3t_2^2 + t_2^3 - xt_2 + \frac{1}{3}x^2t_2 - xt_2^2 - \frac{1}{27}x^3 = 0$$

$$F_3(x,t_3) = 3t_3 + 3t_2^2 + t_2^3 - xt_3 + \frac{1}{3}x^2t_3 - xt_3^2 - \frac{1}{27}x^3 = 0$$

$$3a_3 - \frac{1}{27} = 0$$

$$a_3 = \frac{1}{81}$$
Problem 2

Stage 4: In $F_3(x,t_3)$ replace $t_3 = \frac{1}{81}x^3 + t_4$

$$F_3(x,t_3) = 3t_3 + 3t_2^2 + t_2^3 - xt_3 + \frac{1}{3}x^2t_3 - xt_3^2 - \frac{1}{27}x^3 = 0$$

$$F_4(x,t_4) = 3\left(\frac{x^3}{81} + t_4\right) + 3\left(\frac{x^3}{81} + t_4\right)^2 + \left(\frac{x^3}{81} + t_4\right)^3 - x\left(\frac{x^3}{81} + t_4\right) +$$

$$= \frac{1}{3}x^2\left(\frac{x^3}{81} + t_4\right) - x\left(\frac{x^3}{81} + t_4\right)^2 - \frac{1}{27}x^3 = 0$$

$$= 3t_4 + 3t_4^2 + t_4^3 - xt_4 + \frac{x^2t_4}{3} + \frac{2x^3t_4^2}{27} - \frac{2x^4t_4^3}{81} + \frac{x^6t_4}{2187} - xt_4^2 +$$

$$= \frac{x^3t_4^2}{27} - \frac{x_4}{81} + \frac{x^5}{243} + \frac{x^6}{2187} - \frac{x^7}{6561} + \frac{x^9}{531441} = 0$$

$$3a_4 - \frac{1}{81} = 0$$

$$a_4 = \frac{1}{243}$$