The good Christian should beware of mathematicians, and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the Devil to darken the spirit and to confine man in the bonds of Hell.

St. Augustine
De Genesi ad Litteram, Book II, xvii, 37
The Setup

$P$ is the point at which the tangent line to the curve is parallel to the secant $QR$.

Where does the line intersect the parabola?

$$ax^2 = mx + b$$

$$ax^2 - mx - b = 0$$

$$x_1 = \frac{m + \sqrt{m^2 + 4ab}}{2a}, \quad x_2 = \frac{m - \sqrt{m^2 + 4ab}}{2a}$$

Points of Intersection

Now, we can find the points of intersection of the line and the parabola, $Q$ and $R$.

$$Q = (x_1, y_1) \quad \Rightarrow \quad y_1 = ax_1 = \frac{m^2 - m\sqrt{m^2 + 4ab} + 2ab}{2a}$$

$$Q = \left( \frac{m - \sqrt{m^2 + 4ab}}{2a}, \frac{m^2 - m\sqrt{m^2 + 4ab} + 2ab}{2a} \right)$$

$$R = (x_2, y_2) \quad \Rightarrow \quad y_2 = ax_2 = \frac{m^2 + m\sqrt{m^2 + 4ab} + 2ab}{2a}$$

$$R = \left( \frac{m + \sqrt{m^2 + 4ab}}{2a}, \frac{m^2 + m\sqrt{m^2 + 4ab} + 2ab}{2a} \right)$$

Slope of the Tangent Line

Here we will use some Calculus to help us.

The slope of the tangent line is the derivative of the function at the point.

$$\frac{d}{dx}(ax^2) = 2ax = m$$

$$x = \frac{m}{2a}$$

$$P = (p_x, p_y) = (x, ax^2) = \left( \frac{m}{2a}, \frac{m^2}{4a} \right)$$
Calculus Again to the Rescue

Again, Calculus will help us find the area, $A$.

\[
A = \int_{x_1}^{x_2} \left( ax^2 - (mx + b) \right) dx
\]

\[
= \frac{a}{3} x^3 - \frac{m}{2} x^2 + bx \bigg|_{x_1}^{x_2}
\]

\[
A = \frac{(m^2 + 4ab)^{3/2}}{6a^2}
\]
Area of Triangle

It does not look like we can find a usable angle here. What are our options?

(1) Drop a perpendicular from \( P \) to \( QR \) and then use dot products to compute angles and areas.

(2) Drop a perpendicular from \( Q \) to \( PR \) and follow the above prescription.

(3) Drop a perpendicular from \( R \) to \( PQ \) and follow the above prescription.

(4) Use Heron’s Formula.

Area of the Triangle

Using Heron’s Formula:

\[
p = d(Q, R) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}
\]

\[
p = \sqrt{(m^2 + 4ab)(1 + m^2)}
\]

\[
q = d(P, R) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}
\]

\[
q = \frac{\sqrt{(m^2 + 4ab)(4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab})}}{4a}
\]

Now, the semiperimeter is:

\[
s = \frac{p + q + r}{2}
\]

\[
s = \frac{\sqrt{m^2 + 4ab}}{8a}
\]

\[
+ \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}}
\]

\[
+ \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}}
\]
Area of the Triangle

Uh – oh!!!!
Are we in trouble? Heron’s Formula states that
the area is the following product:

\[ K = \sqrt{s(s - p)(s - q)(s - r)} \]

This does not look promising!!

\[ K = \left(\frac{(m^2 + 4ab)^3}{4996a^4}\right)^{1/2} \left(\frac{4\sqrt{1 + m^2} + \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}}}{4\sqrt{1 + m^2} + \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}} - \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}})^{1/2} \times \left(\frac{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}}}{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}} - \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}}}\right)^{1/2} \times \left(\frac{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}}}{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}} - \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}}}\right)^{1/2} \times \left(\frac{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}}}{4\sqrt{1 + m^2} - \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}} - \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}}}\right)^{1/2} \]

and then a miracle occurs ...

\[ K = \frac{(m^2 + 4ab)^3}{64a^4} \]

\[ K = \frac{(m^2 + 4ab)^{3/2}}{8a^2} \]

Note then that:

\[ \frac{(m^2 + 4ab)^{3/2}}{6a^2} = A = \frac{4}{3} K \]
How did Archimedes do this?

Claim: \( \triangle PQR = 8\triangle PQS \)

What do we mean by “equals” here?

What did Archimedes mean by “equals”?

What good does this do?

What is the area of the quadrilateral \( \square QSPR \)?

\[ A = K + \frac{1}{8}K \]
A better approximation

What is the area of the pentelateral □QSPTR?

---

The better approximation

Note that the triangle ΔPTR is exactly the same as ΔQSP so we have that

\[ A_t = K + \frac{1}{8}K + \frac{1}{8}K = K + \frac{1}{4}K \]

---

An even better approximation

---
The next approximation

Let's go to the next level and add the four triangles given by secant lines QS, SP, PT, and TR.

\[
\text{area}(\triangle QZ,S) = \text{area}(\triangle SZ,P) = \frac{1}{8} \text{area}(\triangle QSP)
\]

\[
= \frac{1}{8} \left( \frac{1}{8} K \right) = \frac{1}{64} K
\]

\[
\text{area}(\triangle PZ,T) = \text{area}(\triangle TZ,R) = \frac{1}{8} \text{area}(\triangle PTR)
\]

\[
= \frac{1}{8} \left( \frac{1}{8} K \right) = \frac{1}{64} K
\]

The next approximation

What is the area of this new polygon that is a much better approximation to the area of the sector of the parabola?

\[
A_2 = A_1 + \frac{4}{64} K = K + \frac{1}{4} K + \frac{1}{16} K
\]

The next approximation

What is the area of each triangle in terms of the original stage?

\[
K_3 = \frac{1}{8} K_2 = \frac{1}{8} \left( \frac{1}{8} K_1 \right) = \frac{1}{8} \left( \frac{1}{8} \left( \frac{1}{8} K \right) \right) = \frac{K}{8^3}
\]

What is the area of the new approximation?

\[
A_3 = A_2 + 8K_3 = K + \frac{1}{4} K + \frac{1}{16} K + \frac{1}{64} K
\]
The next approximation

Okay, we have a pattern to follow now.

How many triangles do we add at the next stage?

8

What is the area of each triangle in terms of the previous stage?

\[ K_3 = \frac{1}{8} K_2 \]

The next approximation

What is the area of the next stage?

We add twice as many triangles each of which has an eighth of the area of the previous triangle. Thus we see that in general,

\[ A_n = K + \frac{1}{4} K + \frac{1}{16} K + \cdots + \frac{1}{4^n} K \]

This, too, Archimedes had found without the aid of modern algebraic notation.

The Final Analysis

Now, Archimedes has to convince his readers that “by exhaustion” this “infinite series” converges to the area of the sector of the parabola.

Now, he had to sum up the series. He knew

\[ 1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n} + \cdots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \]
The Final Analysis

Therefore, Archimedes arrives at the result

\[ A = \frac{4}{3}K \]

Note that this is what we found by Calculus.

Do you think that this means that Archimedes knew the “basics” of calculus?

Proof of the Claim

Let \( M \) be the midpoint of \( QR \).

Claim 1: The \( x \)-coordinate of \( M \) is the same as that of \( P \), the vertex.

Proof of the Claim

From our coordinate geometry we find that the \( x \)-coordinate of \( M \) is given by:

\[
M_x = \frac{x_1 + x_2}{2}
= \frac{1}{2} \left( \frac{m + \sqrt{m^2 + 4ab}}{2a} + \frac{m - \sqrt{m^2 + 4ab}}{2a} \right)
= \frac{m}{2a}
\]
Claim 2: If \( P \) is a vertex and \( M \) is any point on the chord, then the ratio \( QM^2/PM \) is independent of \( M \).

\[
M_x = \frac{m}{2a} \quad M_y = \frac{y_1 + y_2}{2} = \frac{m(x_1 + x_2) + 2b}{2} = \frac{m^2}{2a} + b
\]

Proof of the Claim

Now the square of the length \( QM \) is

\[
(2QM)^2 = QR^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x_2)^2 + m^2(x_1 - x_2)^2
\]

\[
= (x_1 - x_2)^2(1 + m^2) = \left(2 \frac{m^2 + 4ab}{2a}\right)^2 (1 + m^2)
\]

\[
QM = \frac{(m^2 + 4ab)(1 + m^2)}{4a^2}
\]

Proof of the Claim

Now length \( PM \) is

\[
PM = M_y - P_y = \frac{m^2}{2a} + b - \frac{m^2}{4a}
\]

\[
= \frac{(m^2 + 4ab)}{4a}
\]

\[
\frac{QM^2}{PM} = \frac{(m^2 + 4ab)(m^2 + 1)}{4a^2} = \frac{(m^2 + 1)}{a}
\]
Proof of the Claim

This ratio depends only on the slope of the line and the coefficient of the parabola, not on anything else. Thus it is independent of the point along the chord.

Proof of the Claim

Claim 3: If \( QR \) is a chord of a parabola, \( M \) its midpoint and \( N \) the midpoint of \( MR \). Drop perpendiculars from \( M \) and \( N \) to the \( x \)-axis and let them intersect the parabola at \( P \) and \( T \). Then

\[
PM = \frac{4}{3} NT
\]

Proof of the Claim

Construct \( TW \) parallel to \( MN \). Now from the previous claim we have:

\[
\frac{QM^2}{PM} = \frac{TW^2}{PW}
\]

Since \( MNTW \) is a parallelogram

\[
QM = 2MN = 2TW
\]

\[
QM^2 = 4TW^2 \Rightarrow PM = 4PW
\]

\[
PV = PW + WM \Rightarrow WM = 3PW \Rightarrow PV = \frac{4}{3} WM = \frac{4}{3} NT
\]
Proof of the Claim

Claim 3: $\triangle APQR = 8\triangle PQS$

We know that $Y$ is the midpoint of $PQ$. Thus $SY$ bisects one side of $\triangle QPM$ and is parallel to $PM$. Thus $\triangle QYN$ and $\triangle QPM$ are similar. Also, $SY$ intersects $QM$ at its midpoint and

\[ \frac{YN}{PV} = \frac{1}{4} \quad \frac{SN}{SN} = \frac{2}{3} \quad YN = 2SY \]

\[ \triangle PQN = \triangle PYN + \triangle QYN = 2\triangle PYS + 2\triangle QYS = 2\triangle PQS \]
\[ \triangle PQR = 2\triangle PQM = 4\triangle PQN = 8\triangle PQS \]

Q.E.D.  Quod erat demonstrandum

Quit, enough done.